Gittins Policy on NBUE + DHR(k) Job Sizes

Matthew Maurer

Performance Modeling, 2009
Outline

1. Gittins Policy
   - Gittins Index
   - Gittins Policy Application

2. NBUE + DHR(k) Distributions
   - Gittins Reduction to FCFS + FB(θ)
     - Gittins Index Properties
     - Policy Properties
   - Pareto Example
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   - Gittins Reduction to FCFS + FB(\(\theta\))
     - Gittins Index Properties
     - Policy Properties
   - Pareto Example
Gittins Index Motivation

- K-Armed Bandit Problem
- Optimal Blind Scheduling
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Gittins Index Candidates

- Payoff?
  - Costs not accounted for

- Payoff - Investment?
  - Doesn’t make sense – Payoff and Investment are not necessarily in the same units

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- Maximal Ratio of Payoff to Investment
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Scheduling View of Gittins Index

- We parameterize the Gittins Index over
  - \( a \), the current age of the job
  - \( T \), the service quota

- We can think of varying \( T \) as varying the investment.

\[
J(a, T) = \frac{E[\text{Job Completes}|T]}{E[T_{\text{Completion}}|T]} = \frac{\int_0^T f(a+t)dt}{\int_0^T \bar{F}(a+t)}
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- \( G(a) = \sup_{T \geq 0} J(a, T) \)
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Gittins Policy Motivation

- We are usually blind
- We usually know the distribution, and can approximate it well after some startup time if not
- Optimal!
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Gittins Index Computation

- **Exact**
  - To compute $G(a)$ exactly, we have to compute $J(a, T)$ for some $T$.
  - We need to take the analytic minimum of $J(a, T)$ with respect to $T$.

- **Approximation**
  - We can approximate $J(a, T)$ easily.
  - Optimization of a computationally expensive function over the real line...

- This algorithm was initially developed for discrete time cases, and it shows.
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Gittins Policy Usage

- Generalized Blind Approximation - Impractical
- Specific Distributions - Analytic Simplification
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Problem Statement

- Blind
  - Distribution Head NBUE
  - Distribution Tail DHR after $k$
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To optimize $J$, we calculate its derivative

$$\frac{\delta J}{\delta T} = \frac{f(a+T) \int_0^T \bar{F}(a+t)dt + \bar{F}(a+T) \int_0^T f(a+t)dt}{\int_0^T \bar{F}(a+t)dt}$$

If we let $h$ represent the hazard rate of the distribution, we have

$$\frac{\delta J}{\delta T} = \frac{\bar{F}(a+T)(h(a+T) - J(a,T))}{\int_0^T \bar{F}(a+t)dt}$$
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Lemmas

- We introduce the notation $T_a$ to represent the optimal $T$ choice for a job of age $a$
- We omit the proofs for these Lemmas for time and relevance

  $\forall a, x : a \leq x < a + T_a, G(a) \leq G(x)$

  $\forall a : T_a < \infty, G(a + T_a) \leq G(a)$
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Proof Overview

- \( T_0 \geq k \)
- \( \forall a : a < T_0, G(a) \geq G(0) \)
- \( \forall a : a > k, G(a) \) is decreasing
- \( \forall T_0 : T_0 < \infty, G(T_0) \geq G(0) \)
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Property I

- Take some \( x : 0 < x < k \)
- As it has a NBUE head, \( H(x) \geq H(0) \)
- Converting to \( J \), \( J(x, \infty) \geq J(0, \infty) \)
- \( \frac{F(x)}{\int_x^\infty F(t) dt} \geq \frac{1}{\int_0^\infty F(t) dt} \)
- Running math, we get \( \frac{1}{\int_0^\infty F(t) dt} \geq \frac{F(x)}{\int_0^x F(t) dt} \)
- Back in index form, this gives \( G(0) \geq J(0, x) \)
- As \( x \) is valid from 0 to \( k \), we have \( T_0 \geq k \)
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Property II

See the first lemma. The proof is omitted as it is a sufficiently general result.
Setting our derivative to zero, we get the equation

\[ \frac{\bar{F}(a+T)(h(a+T)-J(a,T))}{\int_0^T \bar{F}(a+t)dt} = 0 \]

Excluding infinite \( T \), the \( \bar{F} \) term will not zero, so we have

\[ h(a+T) = J(a, T) \]

For \( a \geq k \), we have the DHR property, so \( G(a) = J(a, 0) = h(a) \)

We have the DHR property, so \( G(a) \) is decreasing for \( a \geq k \).
Property III

- Setting our derivative to zero, we get the equation
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Property IV

See the second lemma. The proof is omitted as it is a sufficiently general result.
Policy Derivation

- We have $\forall a : a < T_0, G(a) \geq G(0)$ and $\forall T_0 : T_0 < \infty, G(T_0) \leq G(0)$
- So, the Gittins Index passes its starting position at some point.
- We have $\forall a : a > k, G(a)$ is decreasing
- So, the Gittins Index keeps going down after that.
- As we start NBUE, and end with this property, by optimality of Gittins
- FCFS + FB($T_0$)
- Additionally, we have the bound $T_0 > k$
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  - $\text{FCFS + FB}(T_0)$
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Qualification

- Up through $k$, NBUE (starts at zero, then jumps)
- After $k$, DHR
- Fits the requirements for this application of Gittins
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Gittins Index
Summary

- When doing blind scheduling, Gittins Policy is optimal.
- The Gittins Policy is usually intractible.
- In our particular case, Gittins reduces to FCFS + FB($T_0$) for NBUE + DHR($k$).
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Summary

- When doing blind scheduling, **Gittins Policy is optimal**.
- The **Gittins Policy is usually intractible**.
- In our particular case **Gittins reduces to FCFS + FB(\(T_0\)) for NBUE + DHR(k)**.
For Further Reading

M. Pinedo.  

S. Aalto, U. Ayesta.  
Optimal scheduling of jobs with a DHR tail in the M/G/1 queue.  

J. Gittins.  
Bandit Processes and Dynamic Allocation Indices.  
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M. Pinedo.  
*Scheduling: Theory, Algorithms and Systems.*  

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