Curvatures, Invariants and How to Get Them Without (M)Any Derivatives

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Classical Notions

Curves

- arclength parameterization
  \[ c : I \rightarrow \mathbb{R}^3 \quad |\ddot{c}(t)| = \kappa(t) \]
- center of osculating ("kissing") circle (also defines osc. plane)
  - tilt of plane is torsion
- Euclidian motion invariant
  - uniquely characterizes curve!
SURFACES

First fundamental form
- parameterized surface
  \[ S : \mathbb{R}^2 \supset \Omega \rightarrow \mathbb{R}^3 \]
  \[ S(u, v) = (x(u, v), y(u, v), z(u, v)) \]
- tangent vectors
  \[ c : I \rightarrow S \quad c(0) = p \quad \dot{c}(0) = \alpha \]
- tangent space \( T_pS \)

TANGENT VECTOR
Tangent Plane

Normal Vector
Metric

Measure stuff
- angle, length, area
- symmetric, bilinear form \( I_p : T_p S \to \mathbb{R} \)
  \[
  I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0
  \]

Measuring area
- areas in tangent space
  \[
  \int \int_{\Omega} |S_{u} \wedge S_{v}| \, du \, dv = A(S) = \int_{S} 1 \, dA
  \]
- no dependence on parameterization
- discrete setting... easy
- sum areas of triangles
**Metric**

Within each triangle

- the metric is obvious

**Geometry of the Normal**

Gauss map

- normal at point

\[ N(p) = \frac{S_u \wedge S_v}{|S_u \wedge S_v|}(p) \]

- consider curve in surface again
  - study its curvature at p
  - “tilting” of normal along the curve
**Shape Operator**

Derivative of Gauss map  
- tangent space to itself  
  \[ dN_p : T_p S \to T_p S \]  
- second fundamental form  
  \[ II_p(v) = -\left\langle dN_p(v), v \right\rangle \]

**Curvature of Curves**

Normal curvature  
- curve in surface  
  \[ \kappa_n(p) = \kappa(p) \cos \theta \]  
- all curves with same tangent vector have same normal curvature!  
  \[ II_p(\dot{c}(0)) = -\left\langle dN_p(\dot{c}(0)), \dot{c}(0) \right\rangle = -\left\langle \dot{N}(0), \dot{c}(0) \right\rangle = \left\langle N, \dot{c}(0) \right\rangle = \left\langle N, \kappa n \right\rangle(p) = \kappa_n(p) \]
INVAR IA NTS

Mean and Gaussian curvature
- determinant and trace only
  \[ \det dN_p = \kappa_1 \kappa_2 = K \]
  \[ \text{tr} dN_p = -(\kappa_1 + \kappa_2) = -2H \]
- eigen values and (ortho) vectors
  \[ dN_p(e_1) = -\kappa_1 e_1 \quad dN_p(e_2) = -\kappa_2 e_2 \]
  \[ \max \rightarrow \kappa_1 \quad \min \rightarrow \kappa_2 \]
Curvatures

Integral representations

- smooth setting

\[ H_p = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) \, d\theta \]

\[ K_p = \lim_{A \to 0} \frac{A_G}{A} \frac{\int_{S_A} s^2 \, dS^2}{\int_{S_A} dA} = \frac{\int |N_u \wedge N_v|}{\int |S_u \wedge S_v|} \]

Discrete Setup?

CS177 (2011) - Discrete Differential Geometry
GAUSSIAN CURVATURE

On a mesh

- can’t take the limit... \( K_p = \lim_{A \to 0} \frac{A_G}{A} \)
- average does make sense

\[ \int_A K_p \approx A K_p \approx A_G \]

only makes sense as an integral, NEVER pointwise

Discrete Gauss curvature at a vertex

A GOOD DEFINITION?

Gaussian curvature over a surface

- Gauss-Bonnet

\[ 2\pi \chi = \int_S \kappa_1 \kappa_2 dA = \int_S K dA \]

- discrete

\[ K_i = 2\pi - \sum_{ijk} \alpha_{jk} \]

\[ \sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi \chi \]
**A Good Definition?**

Spherical Vertex \(2\pi - \Sigma \theta_i > 0\)

Euclidean Vertex \(2\pi - \Sigma \theta_i = 0\)

Hyperbolic Vertex \(2\pi - \Sigma \theta_i < 0\)

developable...

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**Scalar Mean Curvature**

Integral representation

- variation along a vector field

\[
\partial_V \text{Vol} = \int \langle V, N \rangle \, dA
\]

\[
\partial_V \text{Area} = \int \langle V, H \rangle \, dA
\]

\[
H = \frac{\partial_V \text{Area}}{\partial_V \text{Vol}}
\]

\[
\frac{\int H \, dA}{\int N \, dA} \rightarrow H_p
\]

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Boundary Integrals

Vector area

- volume gradient: vector area
  \[ D \subset S \quad \gamma = \partial D \]
  \[ \int_D N \, dA = 1/2 \int_{\gamma} S \times dx = A_{\gamma} \]

- discrete version
  \[ 3A_i = 1/2 \sum p_j \times p_{j+1} \]
  only makes sense as an integral, NEVER pointwise
  area weighted triangle normals

Area gradient

- vector mean curvature
  \[ D \subset S \quad \gamma = \partial D \]
  \[ \int_D H \, dA = \int_{\gamma} N \times dx \]

- discrete version
  \[ H_e = e \times N_1 - e \times N_2 \]
  \[ |H_e| = |e| 2 \sin \theta / 2 \]
  only makes sense as an integral, NEVER pointwise
**Boundary Integrals**

**Area gradient**
- Vector mean curvature
  \[ D \subset S \quad \gamma = \partial D \]
  \[ \int_D \mathbf{H} dA = \int_{\gamma} N \times dx \]
- Discrete version
  \[ 2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\mathbf{V}_i A \]
  \[ = \sum_j (\cot \alpha_{ij} + \cot \alpha_{j'i})(p_i - p_j) \]

*only makes sense as an integral, NEVER pointwise*

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**Laplace-(Beltrami)**

**Surface over tangent plane**
- In eigen basis
  \[ H_p = \Delta S = \left( \frac{d^2}{du^2} + \frac{d^2}{dv^2} \right) S \]
- Laplace-Beltrami
  \[ \mathbf{H} = \Delta S \]

*principal curvature directions*
STEINER POLYNOMIAL

And now for a totally different view

- consider convex polyhedron
- Steiner: \( \text{Vol}(N_t(P)) = \text{Vol}(P) \)
  \[ + t \text{ Area}_P + \frac{t^2}{2} \int_P 2H \, dA + \frac{t^3}{3} \int_P K \, dA \]
- vertices?

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MINIMAL SURFACE

Minimum area energy
- minimal surface
  \[ E_A = \int_S 1 \, dA \]

CS177 (2011) - DISCRETE DIFFERENTIAL GEOMETRY
Minimal Surface

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**Minimal Surface**

Minimum area energy
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Minimal Surface

Minimum area energy
- minimal surface

$$E_A = \int_S 1 dA$$

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**Minimal Surface**

Minimum area energy
- minimal surface

$$E_A = \int_S 1\,dA$$

$$2\partial_i A_{ijk} - R^g/2(p_k - p_j)$$

$$\sum_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j) = 0$$

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**Mean Curvature Flow**

Laplace-Beltrami
- Dirichlet energy

$$\min \int (\nabla u)^2 \Rightarrow \Delta u = 0$$

$$u|_{\partial\Omega} = u_0$$

- on surface

$$H = \Delta_S S = \frac{\nabla A}{2A}$$

$$\partial_i p_i = -H_i/2A_i$$

$$= -1/4A_i \sum_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)$$
Mean Curvature Flow

Laplace-Beltrami

- Dirichlet energy

\[ \min \int (\nabla u)^2 \sim \Delta u = 0 \quad u|_{\partial \Omega} = u_0 \]

- on surface

\[ H = \Delta_S S = \frac{\nabla A}{2A} \]

\[ \partial_t p_i = -H_i/2A_i \]

\[ = -1/4A_i \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j) \]
Mean Curvature Flow

Laplace-Beltrami

- Dirichlet energy

\[
\min \int (\nabla u)^2 \quad \Delta u = 0 \\
\left. u \right|_{\partial \Omega} = u_0
\]

- on surface

\[
\partial_i p_i = -H_i/2A_i \\
= -1/4A_i \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji}) (p_i - p_j)
\]

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**Mean Curvature Flow**

Laplace-Beltrami
- Dirichlet energy
  \[ \min \int (\nabla u)^2 \Rightarrow \Delta u = 0 \]
  \[ u|_{\partial \Omega} = u_0 \]
  \[ \text{on surface} \]
  \[ \mathbf{H} = \Delta_S S = \frac{\nabla A}{2A} \]
  \[ \partial_t p_i = -\mathbf{H}_i/2A_i \]
  \[ = -1/4A_i \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j) \]

**Convergence?**

Can be tricky...
- see Cohen-Steiner paper
- think about chinese lanterns...
  - Schwarz’s example a good one to keep in mind
Parameterizations

What is a parameterization?
- function from some region $\Omega \subset \mathbb{R}^2$ to the embedded surface $M \subset \mathbb{R}^3$

$$S(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

- we go the *other* way around
- how to measure distortion?

Measuring Distortion

Dirichlet energy of a map [Pinkall/Polthier ‘93]
- harmonic param: $\Delta_S u = 0 \quad u|_{\partial \Omega} = u_0$

$$E_D(u) = \int_S (\nabla_S u)^2 \, dA$$

- minimizer is discrete harmonic
  $$0 = \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji}) (u_i - u_j)$$

- need to fix boundary
Harmonic Map

Properties of minimizer

- link with area of triangle??

\[
\left\langle \frac{\partial s}{\partial u_1}, \frac{\partial s}{\partial u_2} \right\rangle = 0
\]

\[
A = \frac{1}{2} \int_T \left| \frac{\partial s}{\partial u_1} \times \frac{\partial s}{\partial u_2} \right| \, du
\]

\[
\leq \frac{1}{2} \int_T \left| \frac{\partial s}{\partial u_1} \right| \left| \frac{\partial s}{\partial u_2} \right| \, du
\]

\[
\leq \frac{1}{4} \int_I \left( \frac{\partial s}{\partial u_1} \right)^2 + \left( \frac{\partial s}{\partial u_2} \right)^2 \, du
\]

Discrete Conformal

Minimizer of conformal energy

- \( E_C(u) = E_D(u) - \text{Area} (\text{Range}(u)) \)

\nabla E_C(u) = 0

Fixed boundary: Dirichlet condition

Free boundary: match gradient of area
**Little Aside**

What’s the param of a flat mesh?
- itself...

**Notion of Barycentric Coordinates**
- vertex can be “reconstructed” by a linear combo of its neighbors
- try it on cot formula...
- can you think of other weights?

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**Recap**

Invariants as overarching theme
- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions
  - variational formulations
  - careful: generally only as averages
**Tools**

Operators we have now
- volume gradient: notion of normal
- area gradient: notion of normal
  - also: mean curvature
- smoothing, parameterization, editing (bi-Laplace-Beltrami)

**Down the Line**

Approach so far
- essentially linear: PL mesh...
- same equations can be derived with
  - DEC: discrete exterior calculus
    - coming right up!