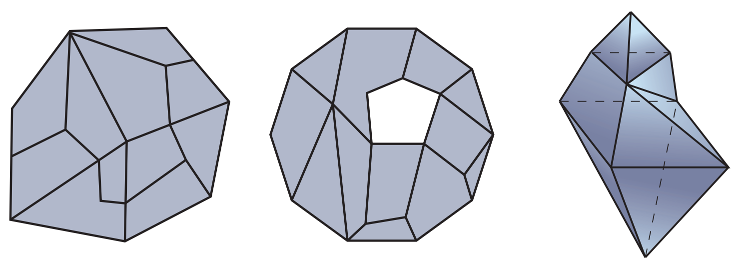
CS 177 - Fall 2010

Homework 1: Topological Invariants of Discrete Surfaces

1 Euler Characteristic

1.1 Polyhedral Formula

A topological *disk* is, roughly speaking, any shape you can get by deforming the region bounded by a circle without tearing it, puncturing it, or gluing its edges together. Some examples of shapes that are disks include the interior of a square, a flag, a leaf, and a glove. Some examples of shapes that are *not* disks include a circle, a ball, a sphere, a DVD, a donut, and a teapot. A *polygonal disk* is simply any disk constructed out of simple polygons. Similarly, a topological *sphere* is any shape resembling the standard sphere, and a *polyhedron* is a sphere made of polygons. More generally, a *piecewise linear* surface is any surface made by gluing together polygons along their edges; a *simplicial surface* is a special case of a piecewise linear surface where all the faces are triangles. The *boundary* of a piecewise linear surface is the set of edges that are contained in only a single face (all other edges are shared by exactly two faces). For example, a disk has a boundary whereas a polyhedron does not. You may assume that surfaces have no boundary unless otherwise stated.



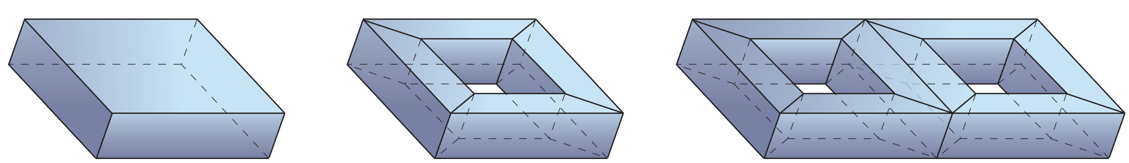
polygonal disk *(neither)* *polyhedron*

Show that for any polygonal disk with V vertices, E edges, and F faces, the following relationship holds:

$$V - E + F = 1$$

and conclude that for any polyhedron $V - E + F = 2$.
(Hint: use induction.)

1.2 Euler-Poincaré Formula



sphere *(g=0)* *torus* *(g=1)* *double torus* *(g=2)*

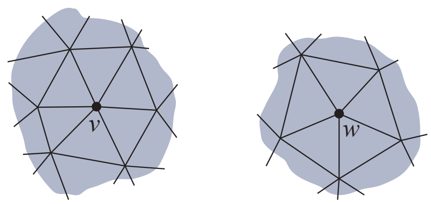
Clearly not all surfaces look like disks or spheres. Some surfaces have additional *handles* that distinguish them topologically; the number of handles g is known as the *genus* of the surface (see illustration above for examples). In fact, among all surfaces that have no boundary and are connected (meaning a single piece), compact (meaning contained in some ball of finite size), and orientable (having two distinct sides), the genus is the *only* thing that distinguishes two surfaces. A more general formula applies to such surfaces, namely

$$V - E + F = 2 - 2g,$$

which is known as the *Euler-Poincaré formula*. (You do not have to prove anything about this statement, but it will be useful in later calculations.)

2 Tessellation

2.1 Regular Valence



Vertex v is regular. Vertex w is irregular.

The *valence* of a vertex in a piecewise linear surface is the number of faces that contain that vertex. A vertex of a *simplicial* surface is said to be *regular* when its valence equals six. Show that the only (connected, orientable) simplicial surface for which every vertex has regular valence is a torus ($g = 1$). You may assume that the surface has finitely many faces.

(Hint: apply the Euler-Poincaré formula.)

2.2 Minimally Irregular Valence

Show that the minimum possible number of irregular valence vertices in a (connected, orientable) simplicial surface K of genus g is given by

$$m(K) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2, \end{cases}$$

assuming that all vertices have valence at least three and that there are finitely many faces.

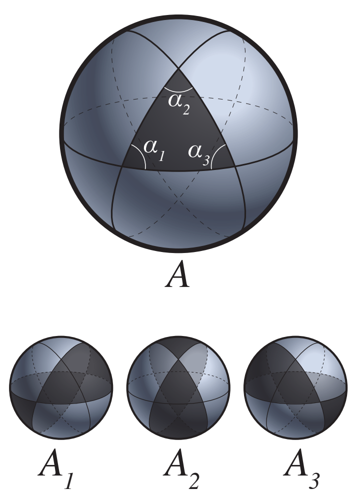
3 Discrete Gaussian Curvature

3.1 Area of a Spherical Triangle

Show that the area of a spherical triangle on the unit sphere with interior angles $\alpha_1, \alpha_2, \alpha_3$ is

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

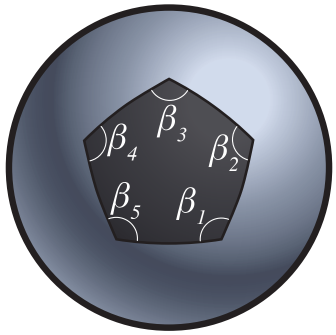
(Hint: consider the areas A_1, A_2, A_3 of the three shaded regions (called "diangles") pictured below.)



3.2 Area of a Spherical Polygon

Show that the area of a spherical polygon with consecutive interior angles $\beta_1, \ldots, \beta_n$ is

$$A = (2-n)\pi + \sum_{i=1}^{n} \beta_i.$$



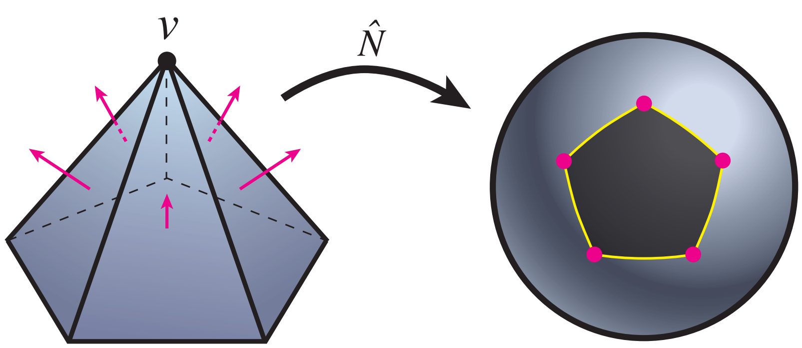
(Hint: use the expression for the area of a spherical triangle you just derived!)

3.3 Angle Defect

Recall that for a discrete planar curve we can define the curvature at a vertex as the distance on the unit circle between the two adjacent normals (see slides from Lecture 1). For a discrete *surface* we can define (Gaussian) curvature at a vertex v as the *area* on the unit sphere bounded by a spherical polygon whose vertices are the unit normals of the faces around v. Show that this area is equal to the *angle defect*

$$d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v)$$

where F_v is the set of faces containing v and $\angle_f(v)$ is the interior angle of the face f at vertex v.
(Hint: consider planes that contain two consecutive normals and their intersection with the unit sphere)



3.4 Discrete Gauss-Bonnet Theorem

Consider a (connected, orientable) simplicial surface K with finitely many vertices V, edges E and faces F. Show that a discrete analog of the Gauss-Bonnet theorem holds for simplicial surfaces, namely

$$\sum_{v \in V} d(v) = 2\pi\chi$$

where $\chi = |V| - |E| + |F|$ is the *Euler characteristic* of the surface.