Thin Shells & Curvature-Based Energy

Thin shells and thin plates

Thin, flexible objects

Shells are naturally *curved*

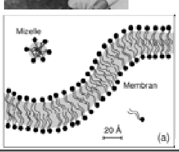
Plates are naturally *flat*

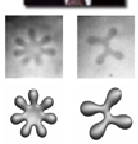
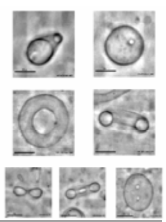


Physics of membranes

S. Helfrich (FU Berlin), P. Canham (U.W. Ontario)





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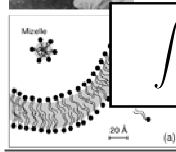
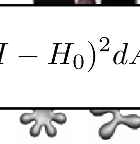
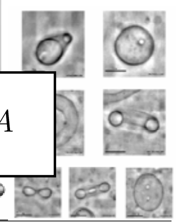
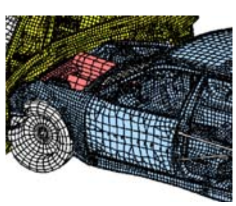


$$\int (H - H_0)^2 dA$$

Engineering

Civil/mechanical/aeronautical design

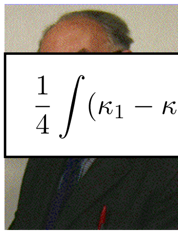



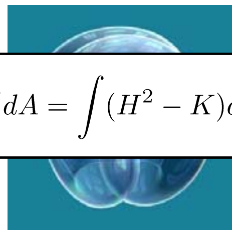
Mathematics

T. J. Willmore's surfaces


Mathematics

T. J. Willmore's surfaces

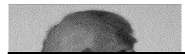




$$\frac{1}{4}\int(\kappa_1-\kappa_2)^2 dA = \int(H^2-K)dA$$

Mathematics

T. J. Willmore's surfaces



topological invariant

$$\frac{1}{4}\int(\kappa_1-\kappa_2)^2 dA = \int(H^2-K)dA$$

$$\int H^2 dA = \frac{1}{4}\int(\kappa_1+\kappa_2)^2 dA$$

Related work

Researchers in graphics:

- Terzopoulos, Bridson, Breen, etc.
  - ad-hoc models for cloth
- Bobenko & Suris, Pai
  - discrete models of elastic curves

[Choi and Ko]

Euler's elastica

Early formulation of elastic curves

$$E^{\text{bend}} = \int_0^l \kappa(s)^2 ds$$

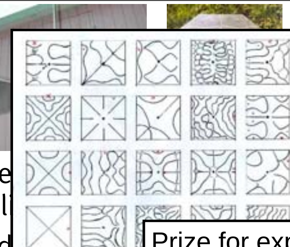
Bernoulli began generalization to surfaces

Chladni's vibrating plates

Plate vibrated by violin bow

Sand settles on nodal curves

Chladni's vibrating plates



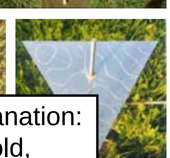


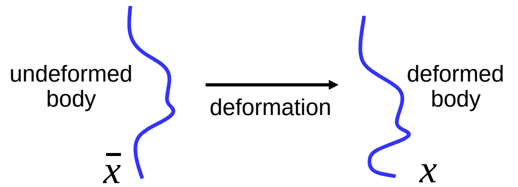
Plate vibrated by violin bow

Sand settles on nodal curves

Prize for explanation:
1kg of gold,
1808,1811,1815

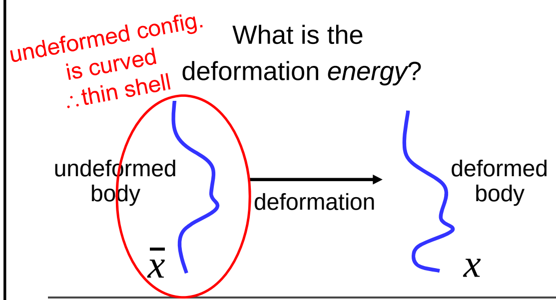
Problem setup

What is the deformation *energy*?



Problem setup

What is the deformation *energy*?



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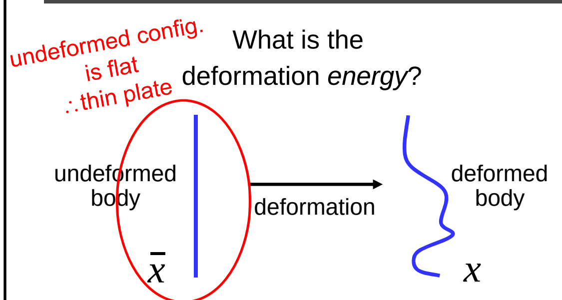
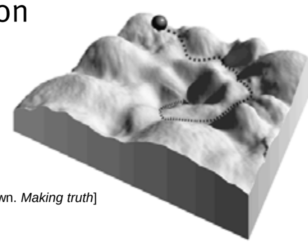


Problem setup

Energy is a non-negative scalar function

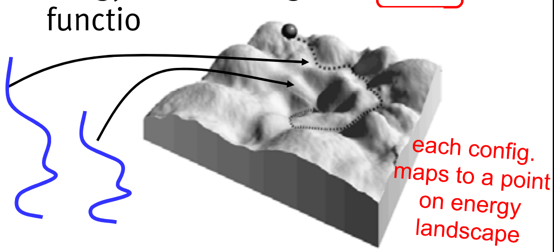


[T. L. Brown. *Making truth*]

Problem setup

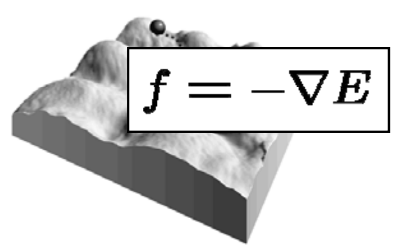
real number,
coordinate-frame invariant

Energy is a non-negative scalar functio



Problem setup

Internal forces push "downhill"



$$f = -\nabla E$$

Plates


Germain


Poisson


Navier

Thin plate energy

Germain's argument:

- bending energy must be a symmetric even fct of principal curvatures

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4}\int (\kappa_1 + \kappa_2)^2 dA$$
$$= \int H^2 dA$$

Thin plate energy

Poisson's linearization

- assuming small displacements, approx. curvature by 2nd derivatives

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4}\int (\kappa_1 + \kappa_2)^2 dA$$
$$E^{bend}_{lin} = \int (\Delta f)^2 dA$$

Thin plate energy

Navier's equation

- to find minimizer for linearized energy, solve a PDE

$$\Delta^2 f = 0$$
$$E^{bend}_{lin} = \int (\Delta f)^2 dA$$

Thin plate energy

Navier's equation

- to
en

$$\partial_{uuuu} f + 2\partial_{uuvv} f + \partial_{vvvv} f$$

$$\Delta^2 f = 0$$
$$E^{bend}_{lin} = \int (\Delta f)^2 dA$$

Axiomatic approach

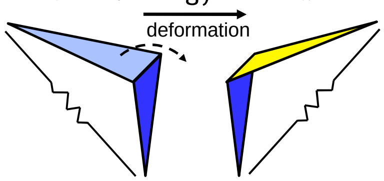
Energy should be:

- symm., even fct of princ. curvatures
- extrinsic measure
- smooth w.r.t. change in shape
- invariant under rigid-body motion
- simple to compute
- easy to understand

What about mass/spring?

Diagonal springs *don't* work

- reference configuration is *curved*
- incorrect energy minima



Axiomatic Shells

"Simplest" answer to desiderata

$$(H - H_0)^2$$

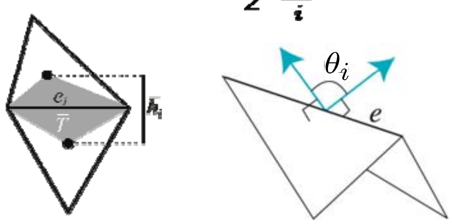
Derivation:

extrinsic change in *shape* operator

$$[\mathrm{Tr}(\varphi^* S) - \mathrm{Tr}(\bar{S})]^2$$

Discrete shells

Elastic energy $= \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$



Discrete shells

Elastic energy $= \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$

Gradient gives forces:

$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla_{\mathbf{x}_k} \theta_i$$

Pimp a cloth simulator

Have a cloth simulator handy?

- reuse all the existing code
- retrofit the bending term
- precompute ref. quantities offline

$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla_{\mathbf{x}_k} \theta_i$$

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new code — reuse code

Modeling Paper

Paper sheet
- curled
- creased
- pinned

Are we done?

Discrete shells are nice and simple.
What else is out there?