Thin Shells & Curvature-Based Energy

Thin shells and thin plates
Thin, flexible objects
Shells are naturally curved
Plates are naturally flat

Physics of membranes
S. Helfrich (FU Berlin), P. Canham (U.W. Ontario)

Mathematics
T. J. Willmore’s surfaces

Engineering
Civil/mechanical/aeronautical design
Mathematics

T. J. Willmore’s surfaces

\[ \frac{1}{4} \int (\kappa_1 - \kappa_2)^2 dA = \int (H^2 - K) dA \]

Related Work

Researchers in graphics:
- Terzopoulos, Bridson, Breen, etc.
- ad-hoc models for cloth
- Bobenko & Suris, Pai
- discrete models of elastic curves

Euler’s elastica

Early formulation of elastic curves

\[ E^\text{bond} = \int_0^L \kappa(s)^2 ds \]

Bernoulli began generalization to surfaces

Chladni’s vibrating plates

Plate vibrated by violin bow
Sand settles on nodal curves

Prize for explanation: 1kg of gold, 1808, 1811, 1815
Problem setup

What is the deformation energy?

undeformed body \( \rightarrow \) deformation \( \rightarrow \) deformed body

\( \vec{x} \) \( \rightarrow \) \( x' \)

Energy is a non-negative scalar function

Internal forces push “downhill”

\( f = -\nabla E \)
Germain’s argument:
- bending energy must be a symmetric even function of principal curvatures

\[ E_{\text{bend}} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA \]

\[ E_{\text{bend}} = \int (\Delta f)^2 dA \]

Poisson’s linearization
- assuming small displacements, approx. curvature by 2nd derivatives

\[ E_{\text{bend}} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA \]

\[ E_{\text{lin}} = \int (\Delta f)^2 dA \]

Navier’s equation
- to find minimizer for linearized energy, solve a PDE

\[ \Delta^2 f = 0 \]

\[ E_{\text{bend}} = \int (\Delta f)^2 dA \]

\[ E_{\text{lin}} = \int (\Delta f)^2 dA \]

Axio-Matic approach
- Energy should be:
  - symm., even fct of princ. curvatures
  - extrinsic measure
  - smooth w.r.t. change in shape
  - invariant under rigid-body motion
  - simple to compute
  - easy to understand
**What about mass/spring?**

Diagonal springs *don’t* work
- reference configuration is *curved*
- incorrect energy minima

---

**Axiomatic Shells**

“Simplest” answer to desiderata

\[
(H - H_0)^2
\]

Derivation:
extrinsic change in *shape* operator

\[
[\text{Tr}(\varphi^* S) - \text{Tr}(\tilde{S})]^2
\]

---

**Discrete Shells**

Elastic energy = \( \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\| \bar{e}_i \|}{h_i} \)

---

**Discrete Shells**

Elastic energy = \( \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\| \bar{e}_i \|}{h_i} \)

Gradient gives forces:

\[
f_k = K_B \sum_i \frac{\| \bar{e}_i \|}{h_i} (\bar{\theta}_i - \theta_i) \nabla x_k \theta_i
\]

---

**Pimp a cloth simulator**

Have a cloth simulator handy?
- reuse all the existing code
- retrofit the bending term
- precompute ref. quantities offline

\[
f_k = K_B \sum_i \frac{\| \bar{e}_i \|}{h_i} (\bar{\theta}_i - \theta_i) \nabla x_k \theta_i
\]
Modeling Paper

Paper sheet
- curled
- creased
- pinned

Are we done?
Discrete shells are nice and simple. What else is out there?

Kirchhoff
Love
Karman
Koiter