Barycentric Coordinates

Interpolation
- given data at sites, interpolate smoothly and intuitively "in between"
- easy over simplices: linear
- more general shapes needed
  - morphing, shape deformation, attribute interpolation, physical modeling, and on and on and on

Basic Principles

Data on boundary; extend
- as affine combination:
  \[ p = \sum_{j=1}^{n} k_j(p) P_j \] then \( f(p) = \sum_{j=1}^{n} k_j(p) f(P_j) \)
- desirables
  - constant precision: \( \sum k_j(x) = 1 \)
  - linear precision: \( \sum k_j(x) x_j = x \)
  - convex: many choices; concave: few…

Basic Setup: Cage

from now on, we’ll focus on Weber/Ben-Chen/Gotsman’s method

Planar Case

Treat everything in complex plane
- tools from complex analysis...
- given source and target polygon
  \{ s_j \}_{j=1,n} \subset \mathbb{C}, \{ f_j \}_{j=1,n} \subset \mathbb{C}, k_j: \mathbb{C} \rightarrow \mathbb{C} \)
  \[ g_{S,F}(z) = \sum k_j(z) f_j \]
- desirables:
  \[ \sum k_j(z) = 1 \quad \sum k_j(z) z_j = z \]

Properties

Complex barycentric interpolation
- preserve similarities
- distinction with real coeffs?
  - preserve affine transformation iff
    \[ \sum \hat{k}_j(z) z_j = z \]
  - not actually desirable… why?
REAL VS. COMPLEX

- Affine transform not quite pleasing...

TRADE-OFF

Must give up something...
- not interpolating anymore

How to find such functions?
- study continuous setting

\[
\int_S k(w,z) \, dw = 1 \quad \int_S k(w,z)w \, dw = z
\]

\[
g_{s,f}(z) = \int_S k(w,z)f(w) \, dw
\]

NOT AN INTERPOLATION

Visualize one coordinate function

\[\Omega\]

CAUCHY FORMULA

Holomorphic functions
- Cauchy kernel: 
  \[
  C(w,z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} \, dw
  \]

- integral version of mean value theorem
  - recover value from average of boundary
- Cauchy coordinates:

\[
g_{s,f}(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(w)}{w - z} \, dw
\]

RESULTING FORMULAS

- apply to polygon
  \[
g_{s,f}(z) = \frac{1}{2\pi i} \sum \int_{\partial S} \frac{f(w)}{w - z} \, dw
\]

- define \( f \) linearly along edge
- grind out integrals...

\[
g_{s,f}(z) = \sum C_j(z) f_j
\]
**Examples**

**Properties**

Best holomorphic function?
- it doesn’t interpolate; is it “best”?  
- closest to given boundary data  
- stick with $C_i$ but use “virtual” poly.  
  $$g_U(z) = \sum C_j(z)u_j$$  
- minimize functional to find best poly.  
  $$\min_U E_S(g_U)$$

**Szego Coordinates**

Optimize “fit”

$$E_S(g_U) = \int_S |g_U(w) - f(w)|^2 dw$$

- need $C_j$ on boundary...  
- define through limit  
- do point collocation (sample boundary)

**Solution**

Pseudo inverse

$$u = C^+ f_S = (C^*C)^{-1} C^* f_S$$

- size is number of vertices (small)  
- letting $H$ be sampling operator:  
  $$f_S = HF$$  
  $$u = C^+ f_S = MF$$  
  $$G_j(z) = \sum_{j=1}^{j=n} C_j(z)M_{i,j}$$

**Example**

**Visualization**

*Source*  

*Szego*  

*Absolute*  

*Real*  

*Imaginary*
Comparison

Analysis

Another Comparison

Visualization

Cauchy-Green vs Szego

Point-to-Point

Simplify UI
- just specify landmarks
- underconstrain (typically)
- add fairness constraint

\[ E_{\text{smooth}}(g) = \int_{S} |g''(w)|^2 dw \]
\[ E_{\text{pts}}^p(g) = \sum_{i=1}^{p} |g(r_i) - f_i|^2 \]

\[ E_{\text{P2P}}^p(g) = E_{\text{pts}}^p(g) + \lambda^2 E_{\text{smooth}}(g) \]

Joint minimization

Point collocation...

\[ E_{\text{P2P}}(g) = \|C_u - f_i\|^2 + \|\lambda D\|^2 \]

Example

P2P Coordinates

Joint minimization

Point collocation...
**Visualization**

- Absolute
- Real
- Imaginary

**Example**

**Video**

Complex Barycentric Coordinates with Applications to Planar Shape Deformation

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