Introduction to

Artificial Intelligence

Lecture 6 – CSPs (cont.)

CS/CNS/EE 154
Andreas Krause

Announcements

- Homework 1 is out. Due Friday Oct 22
- Room for recitation and office hours:
  - Annenberg 107; Tuesday and Thursday 4:30-5:30pm
- Project assignments have been sent out
- Will post details on evaluation soon

- "Science of Iron Man" tonight 8pm (Beckman Auditorium)

Constraint satisfaction problems

- So far: “black box search”
  - Environment state is arbitrary object

- CSPs:
  - state is defined by variables X_i taking values in domain D_i
  - goal test is a set of constraints
  - step cost is 0 – just need to find goal
    (or prove that constraints can’t be satisfied)

- Can develop general purpose algorithms for large class of problems

Example: Map coloring

Variables :

WA, NT, SA, Q, NSW, V, T

Domains:

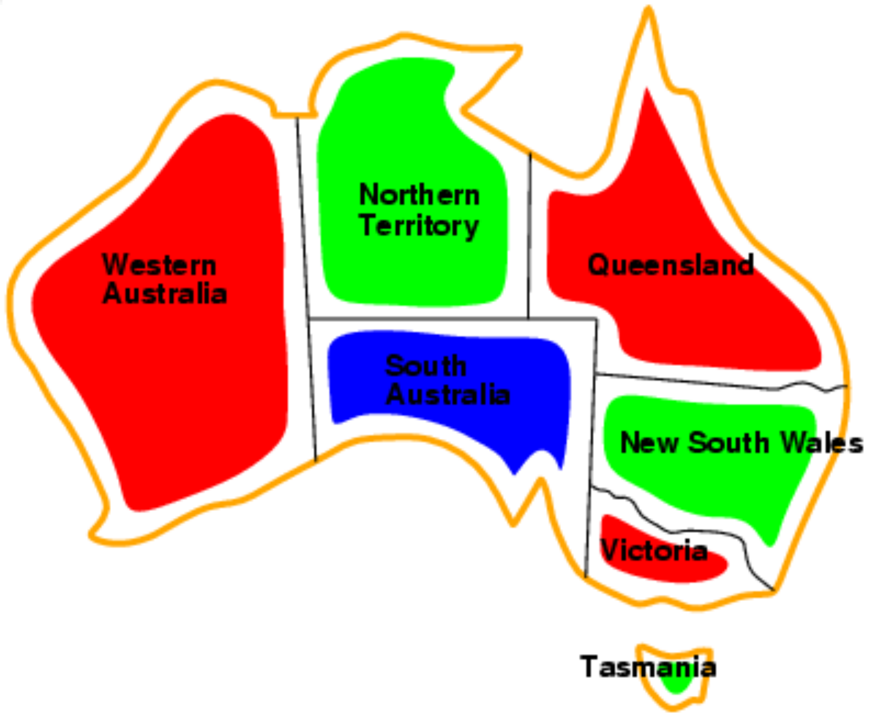
$\{R, B, Y\}$

Constraints

$WA \neq NT \wedge WA \neq SA \wedge \ldots$

$\equiv$

$(WA, NT) \in \{ (R, Y), (R, B), (Y, B), (Y, R)$



- Variables? Domains? Constraints?

Types of CSPs

- Discrete variables
  - Finite domains : Map Coloring, Sudoku, 8 Queens, SAT

    This class

  - Infinite domains : $(X_2 \geq X_1 + 3) \wedge (X_5 \geq X_3 + 2)$

    Job scheduling

    X_i: start time of job i, Domains: $\{1, 2, \ldots\}$

- Continuous variables

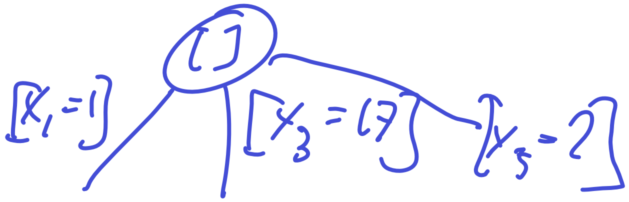
  Robot / factory control, time tabling, ...

Types of constraints

- Unary: involve single variable E.g.: $NSW = B$
- Binary: involve pairs of variables $NSW \neq NT, \ldots$
- Higher-order: involve 3 or more variables
- Soft constraints: violation incurs cost
  - Constraint optimization instead of satisfaction

Solving CSP with search

- Naïve approach
  - State = Partial assignment to variables
  - Successor fn = Assign feasible value to some unassigned var
  - Goal test = check constraints



- Problems?

Size of search tree: $O(n!\, d^n)$

Backtracking search

- Variable assignments are commutative!

[NSW = Y then NT = B] same as [NT = B then NSW = Y]

- Only need to consider assignments to single variable at each node

Size of tree: $O(d^n) \ll n! d^n$

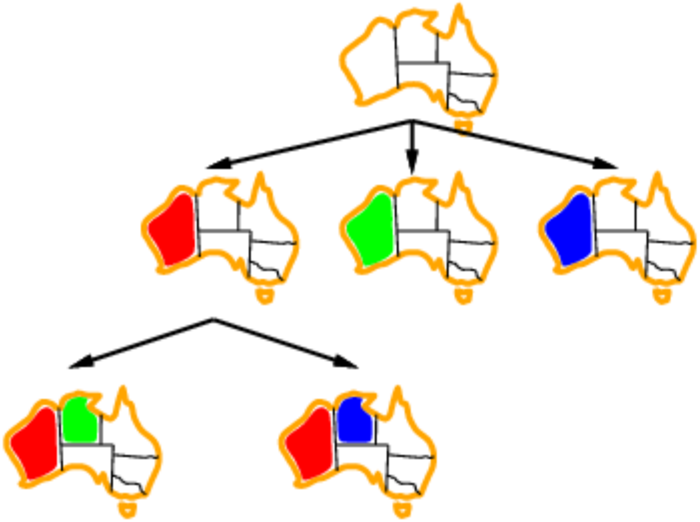
- Depth-first search with single var. assignments is called backtracking search

- Can solve 25-queens

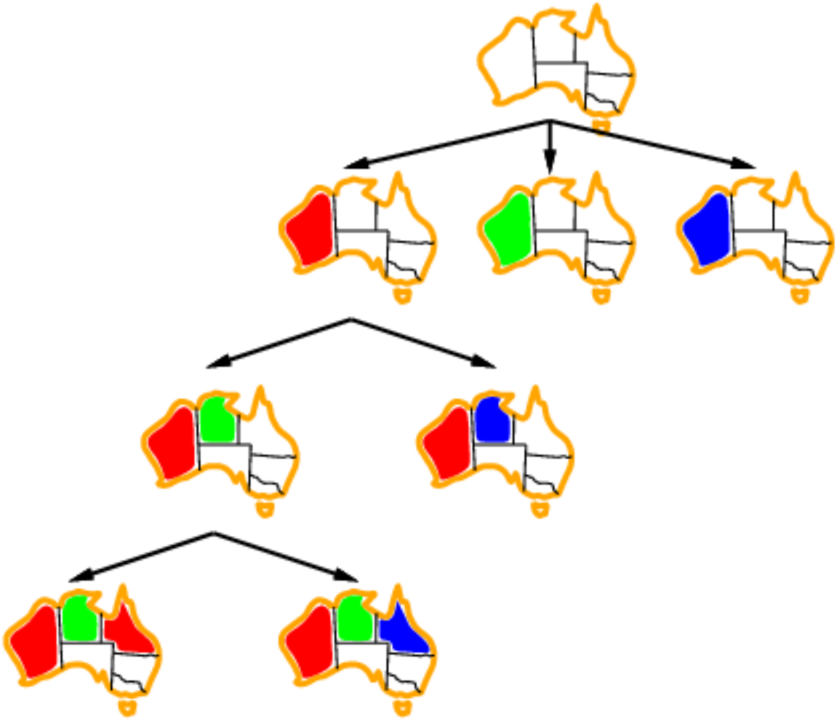
Backtracking example

Backtracking example

Backtracking example



Backtracking example



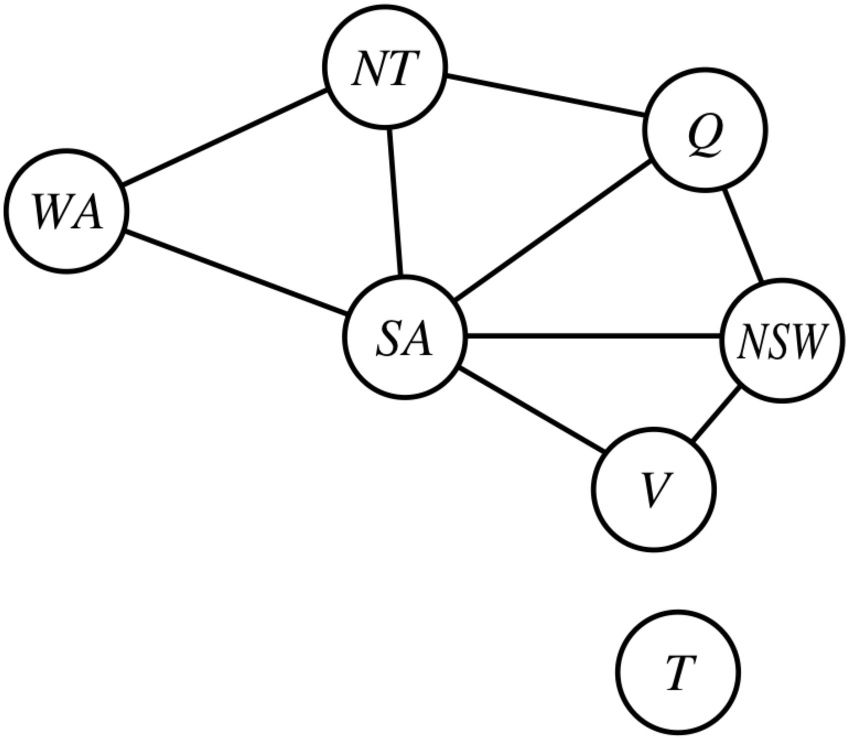
Improving backtracking search

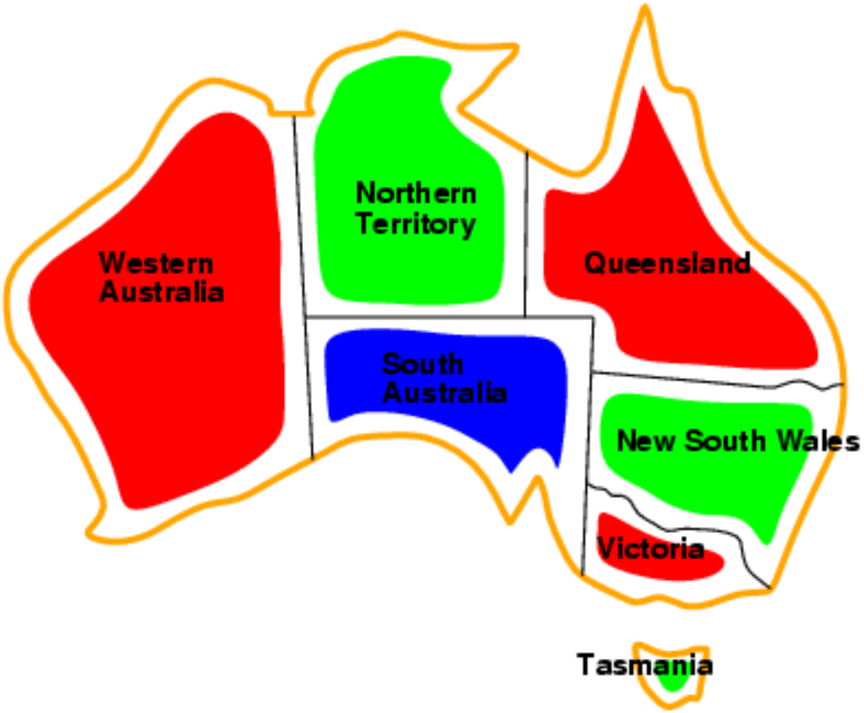
- General purpose methods can drastically improve speed

1. Which variable should be assigned next?
2. In what order should we try the values?
3. Can we detect inevitable failure early?
4. Can we take into account problem structure?

Constraint graph

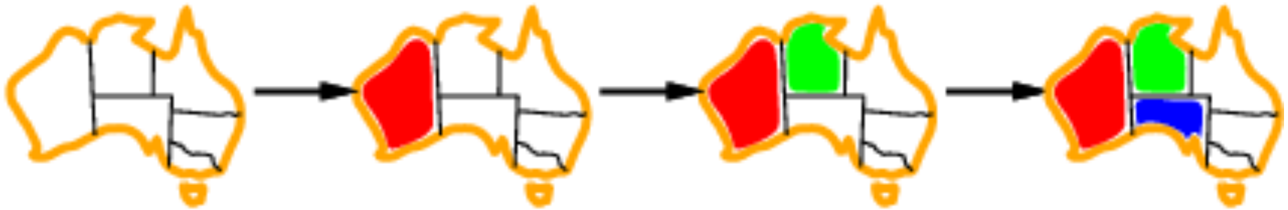
- Nodes: variables
- Arcs: (binary) constraints

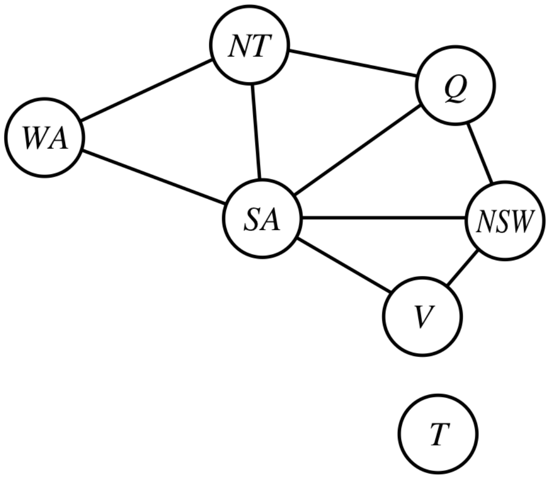




Most constrained variable

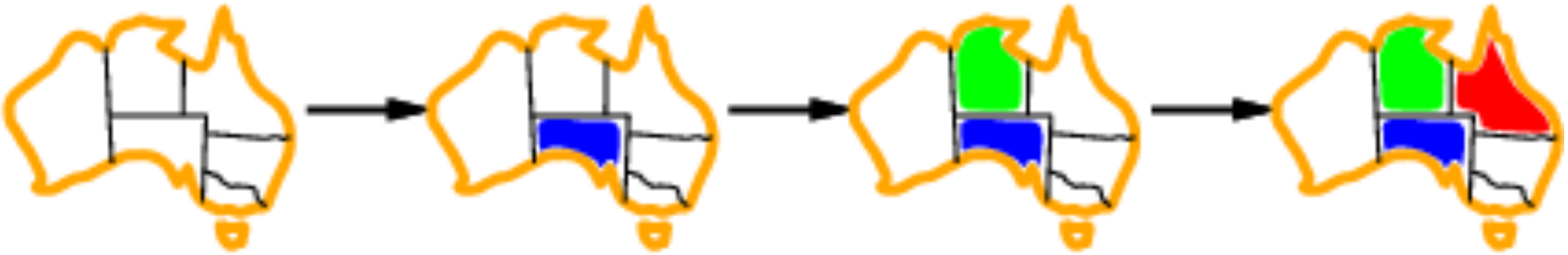
- Most constrained variable:
  - choose the variable with the fewest legal values, a.k.a. minimum remaining values (MRV) heuristic

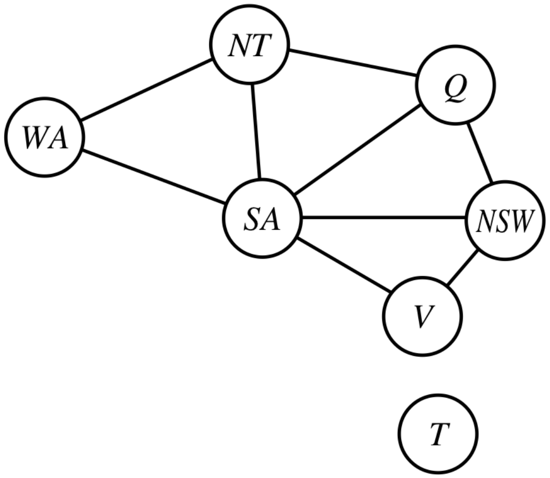




Most constraining variable

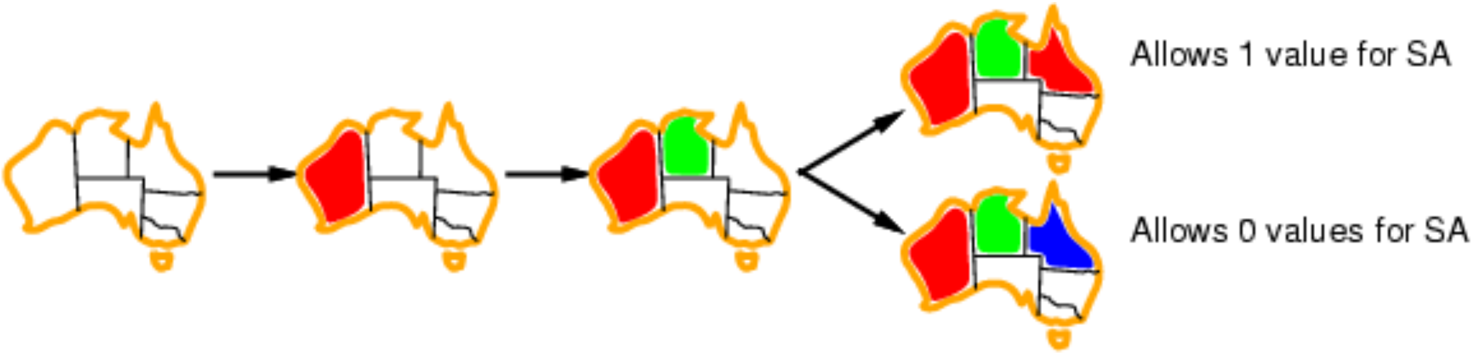
- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables





Least constraining value

- Given a variable, choose the least constraining value (the one that rules out the fewest values in the remaining variables)



- Combining these heuristics makes 1000 queens feasible

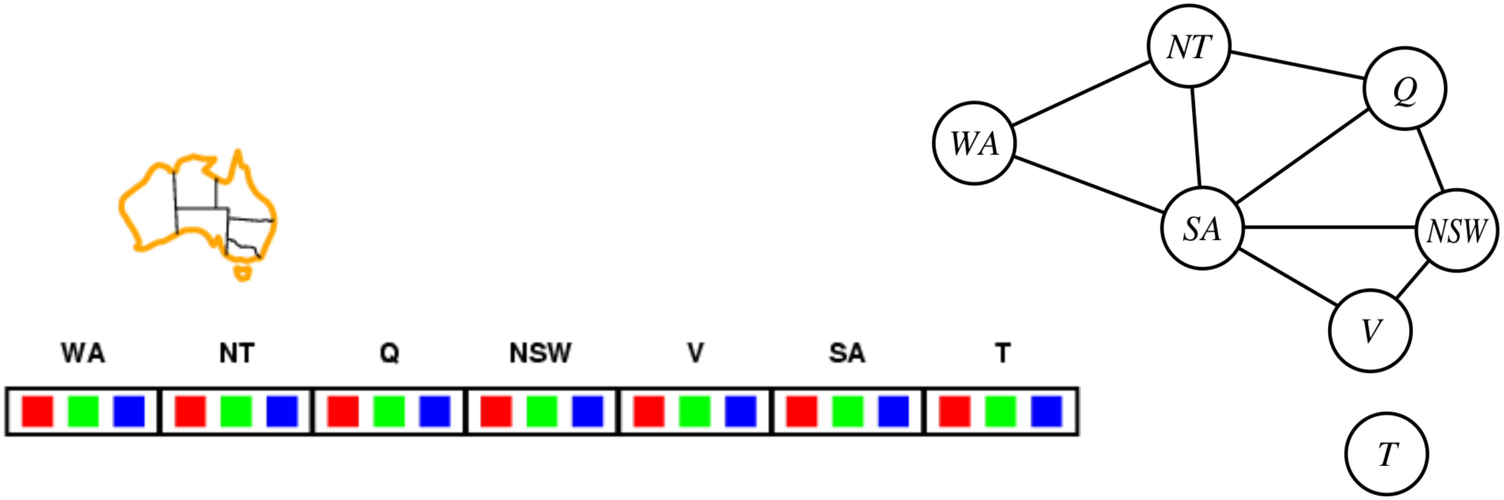
Improving backtracking search

General purpose methods can drastically improve speed

- Which variable should be assigned next?
  - ➔ Most constrained ➔ Most constraining
- In what order should we try the values?
  - ➔ Least constraining
- Can we detect inevitable failure early?
- Can we take into account problem structure?

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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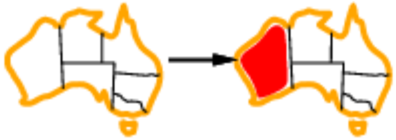

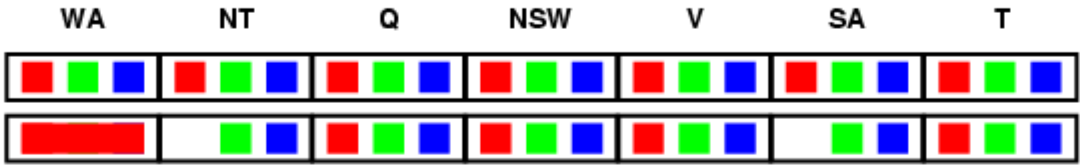

WA	NT	Q	NSW	V	SA	T
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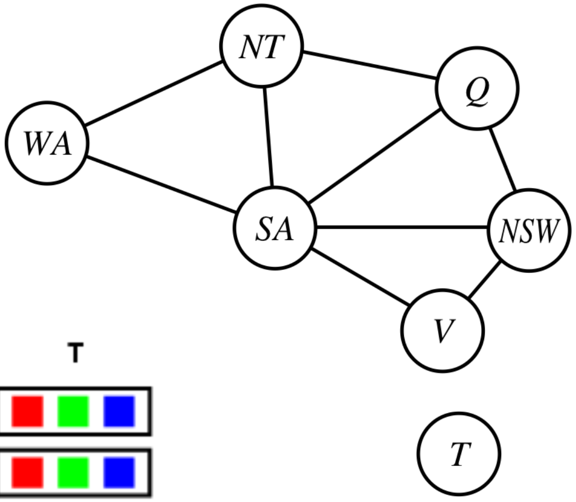

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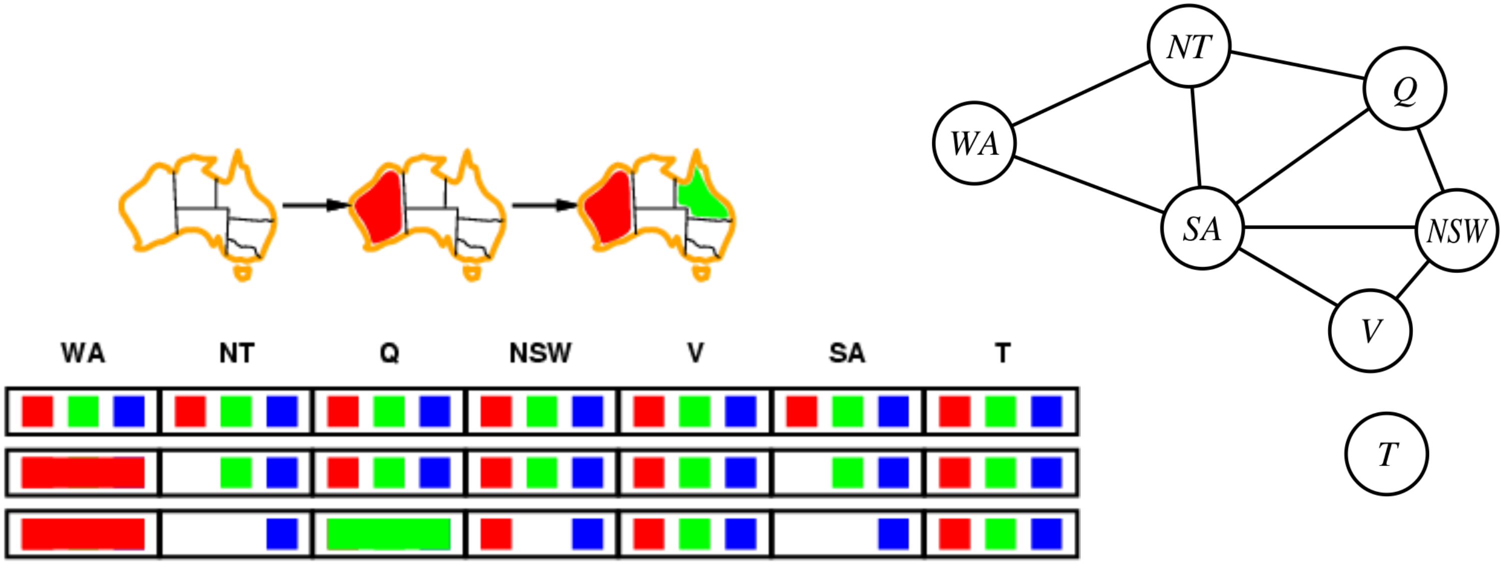

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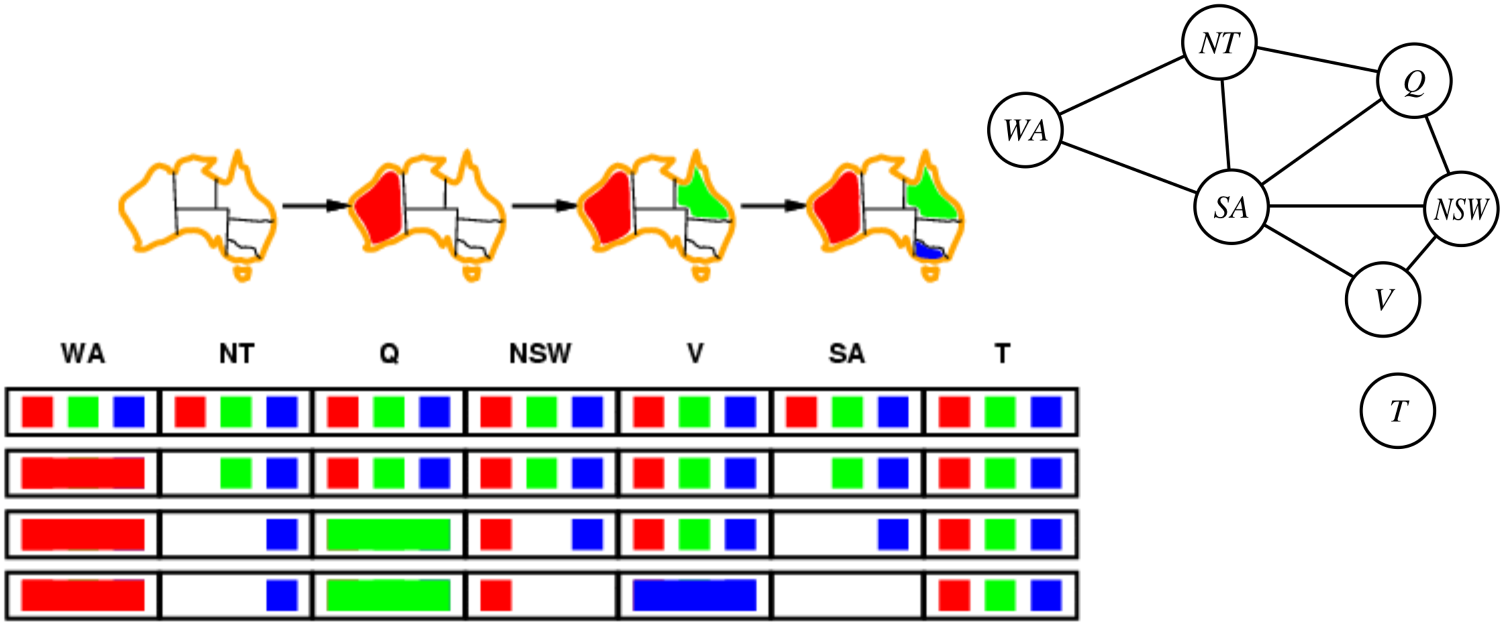

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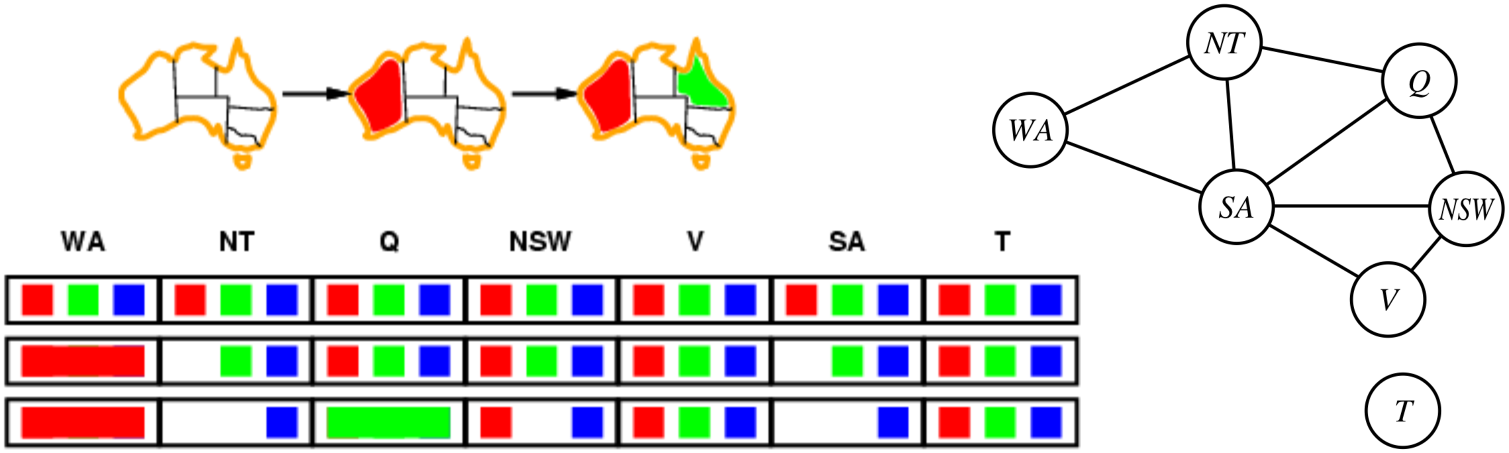
Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



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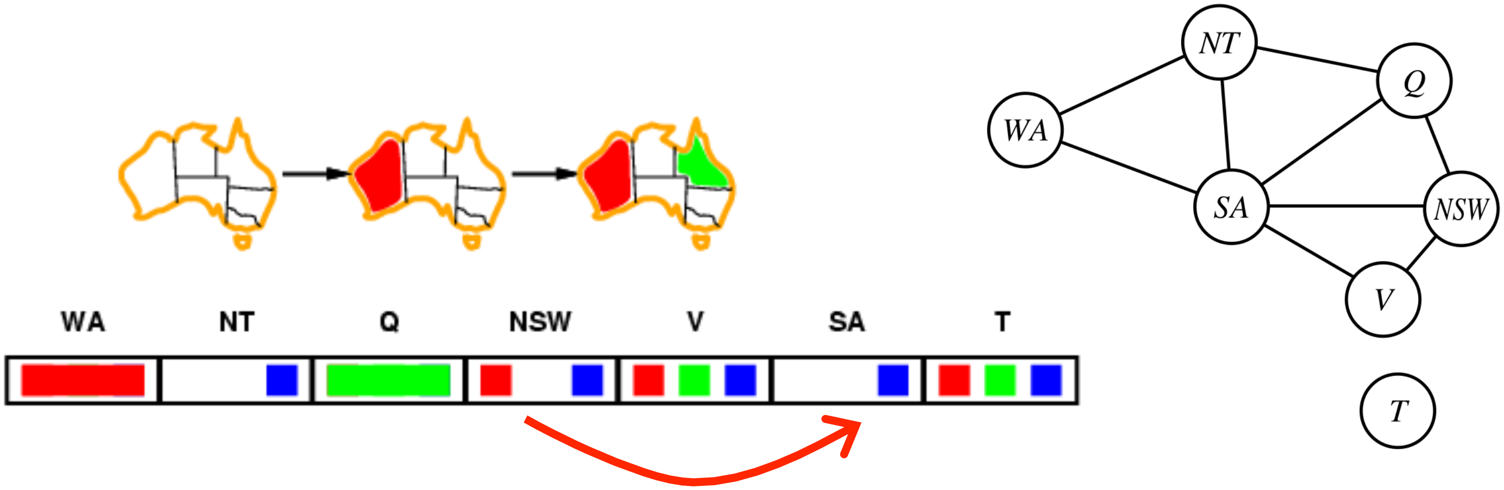
- NT and SA cannot both be blue!
- Can use constraint propagation to detect violations early

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* → *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*



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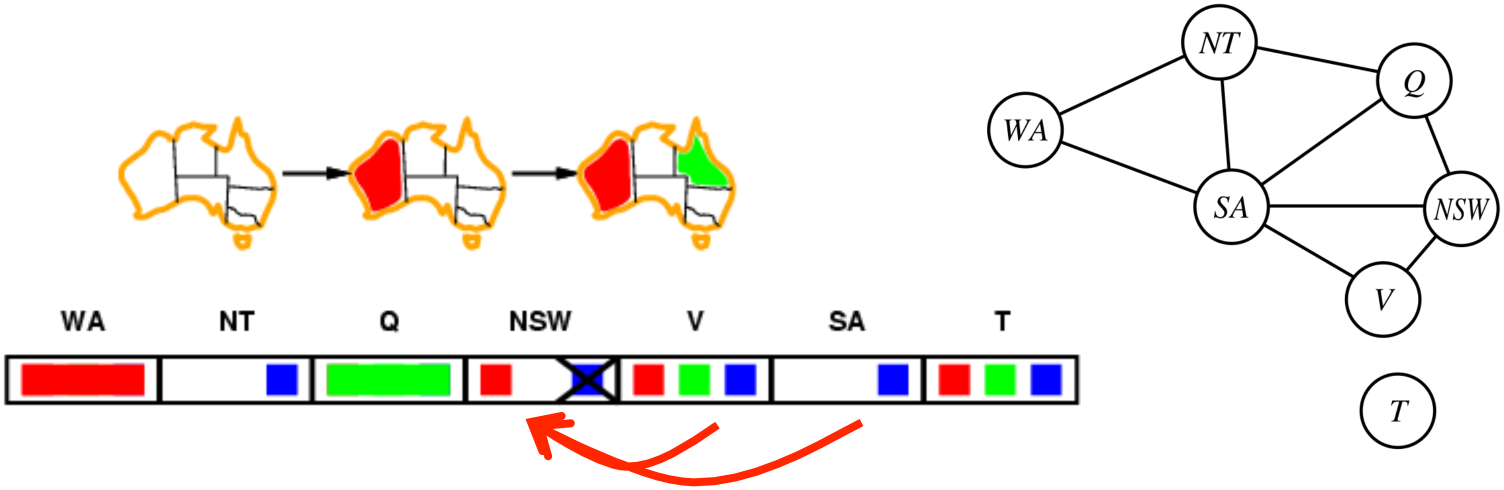

If *X* loses a value, neighbors of *X* need to be rechecked

Arc consistency

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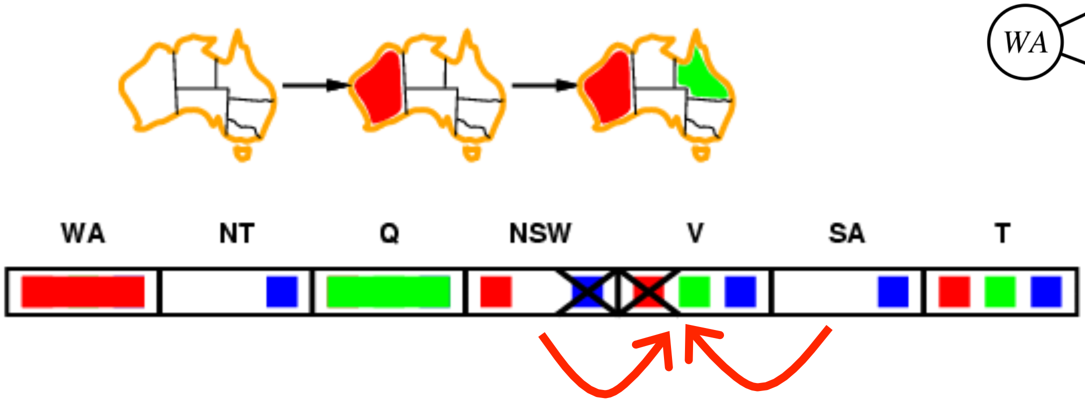
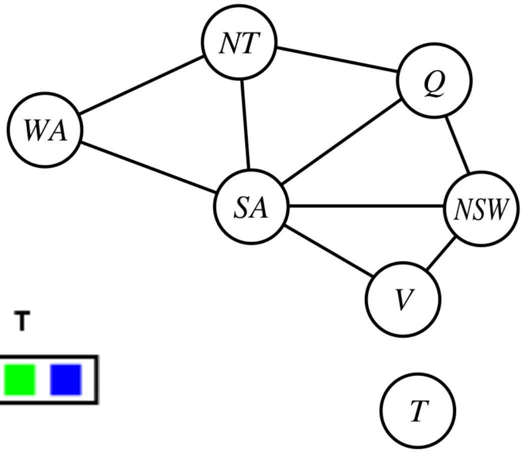


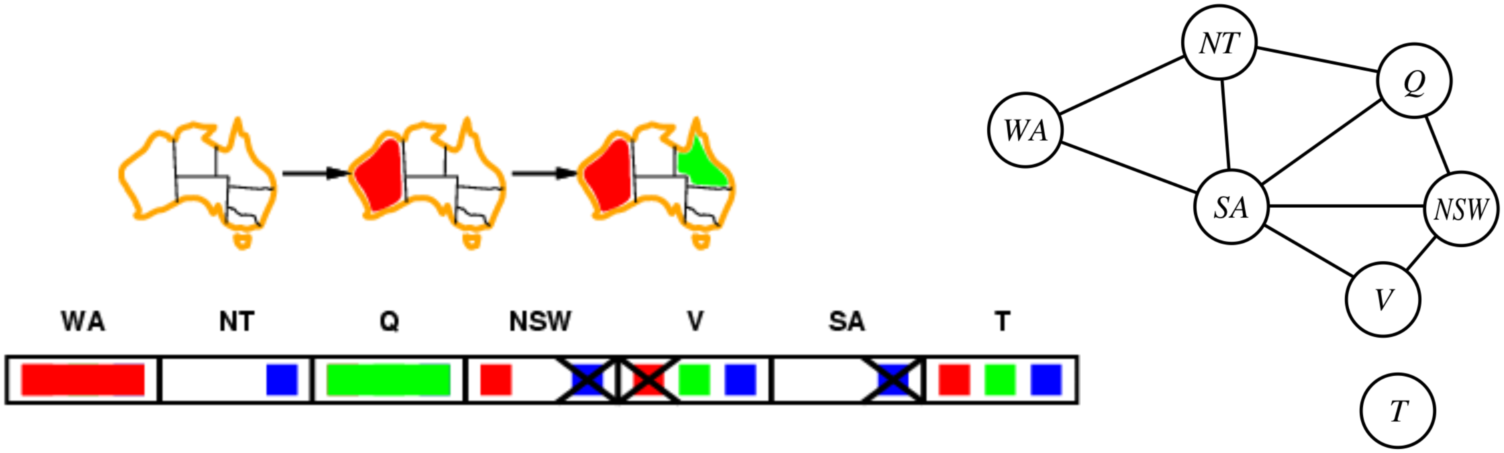


If *X* loses a value, neighbors of *X* need to be rechecked

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y

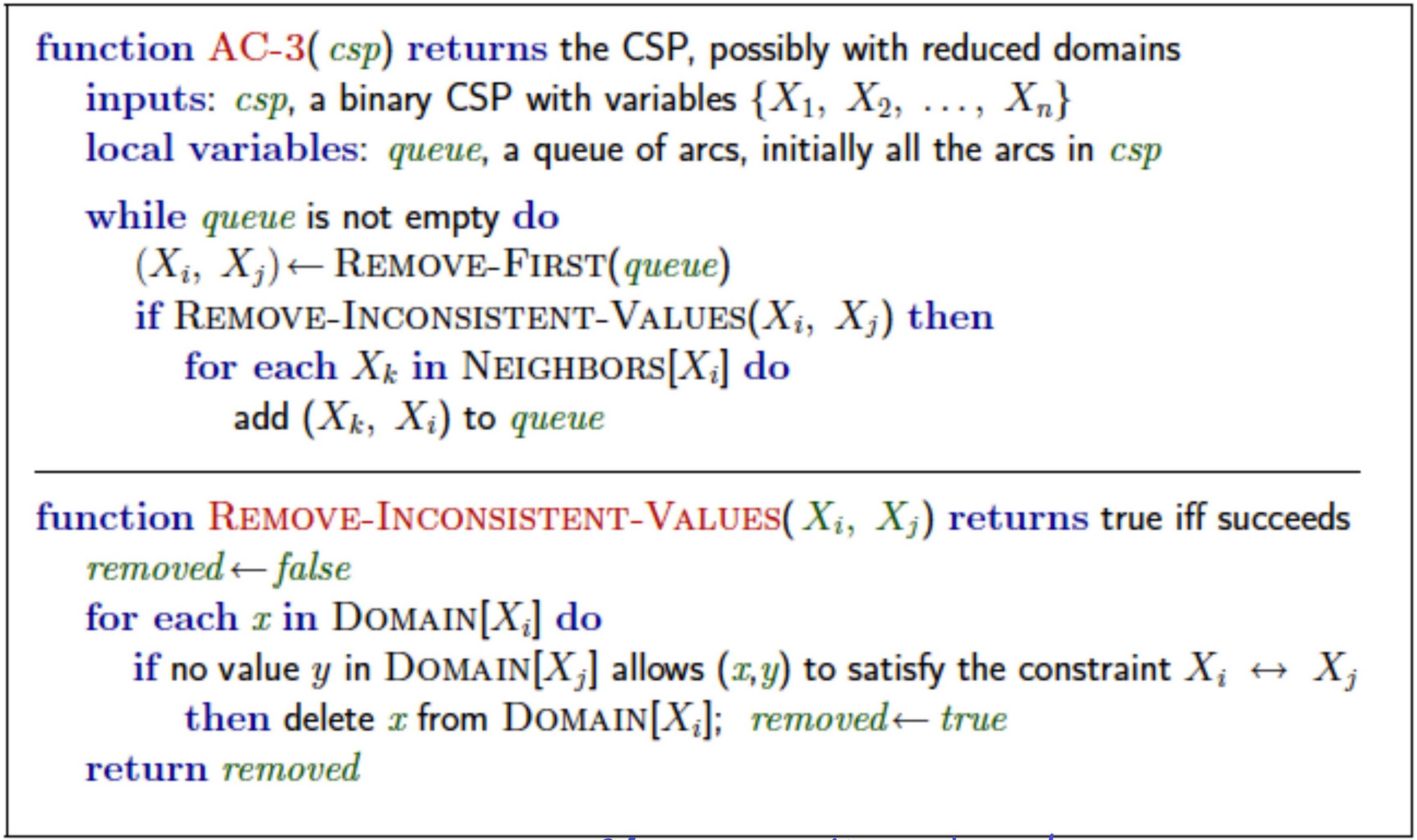


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Only need to recheck arc $X_i \to X_j$ when X_j lost some values

⇒ at most d (:= size of domain (X_j)) times

Arc consistency algorithm AC-3



```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables {$X_1$, $X_2$, ..., $X_n$}
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        ($X_i$, $X_j$) ← REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES($X_i$, $X_j$) then
            for each $X_k$ in NEIGHBORS[$X_i$] do
                add ($X_k$, $X_i$) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i$, $X_j$) returns true iff succeeds
    removed ← false
    for each $x$ in DOMAIN[$X_i$] do
        if no value $y$ in DOMAIN[$X_j$] allows ($x$,$y$) to satisfy the constraint $X_i \leftrightarrow X_j$
            then delete $x$ from DOMAIN[$X_i$]; removed ← true
    return removed
```

Complexity = $O(c \cdot d^3)$

c : # constraints
d : max domain size

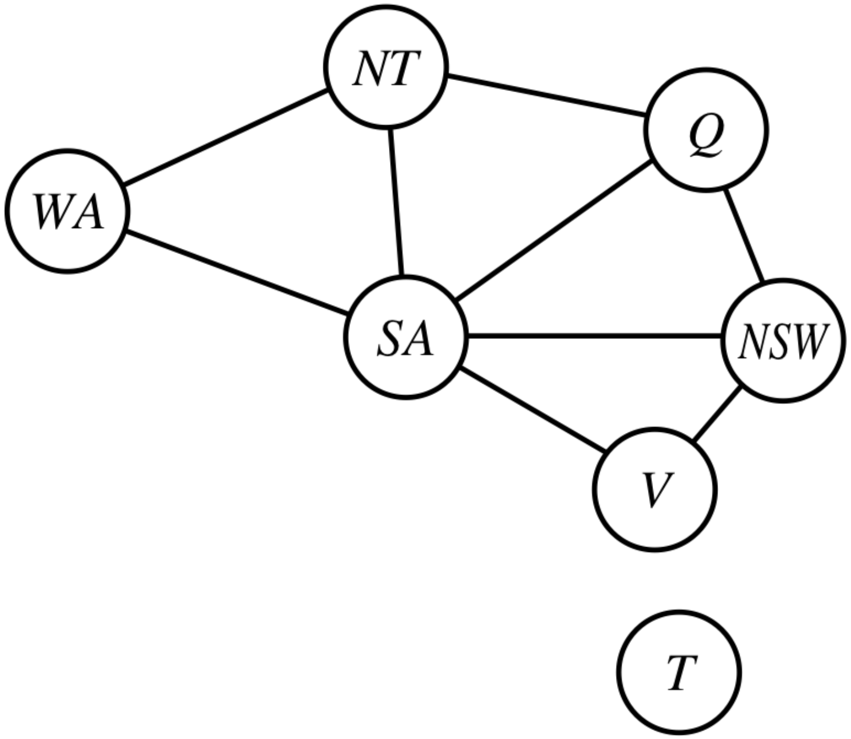
Improving backtracking search

General purpose methods can drastically improve speed

- Which variable should be assigned next?
  - ➔ Most constrained ➔ Most constraining
- In what order should we try the values?
  - ➔ Least constraining
- Can we detect inevitable failure early?
  - ➔ Forward checking, constraint propagation
- Can we take into account problem structure?

Problem structure

- Constraint graph

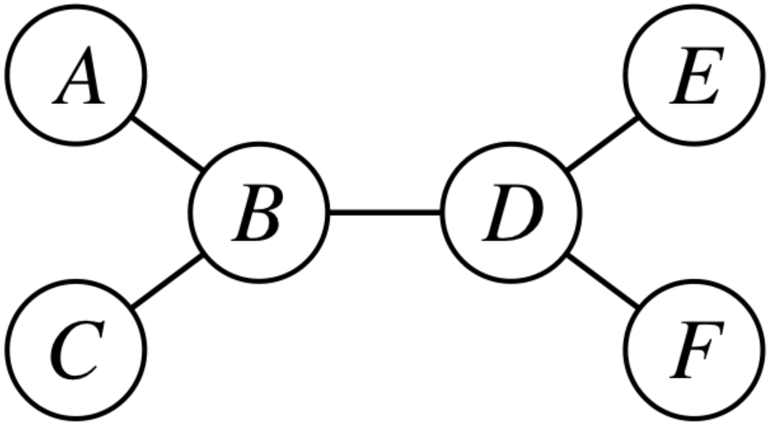


- Suppose we have n variables, grouped into indep. Subproblems with at most c variables

Size of search tree:

$\frac{n}{c} \cdot d^c \ll d^n$

Tree structured CSPs

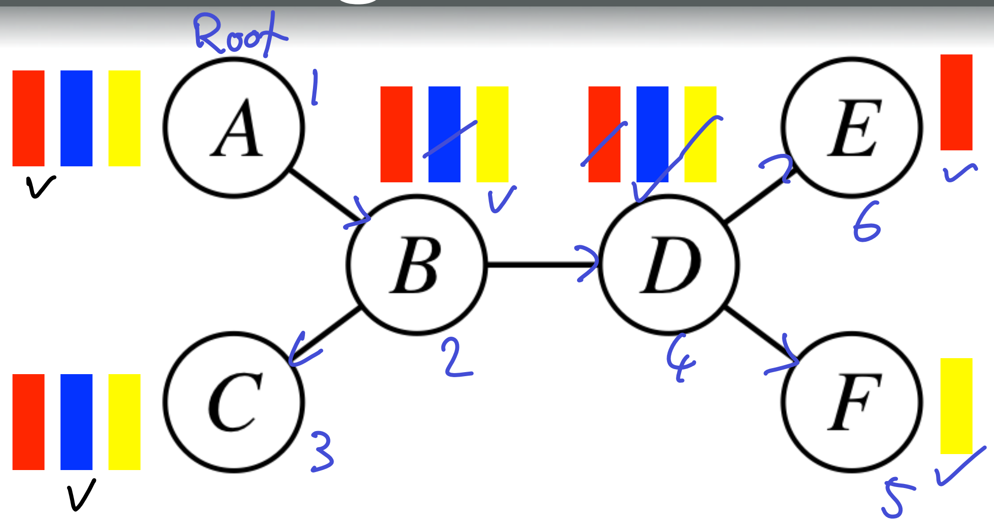


Theorem: If CSP has tree structure, can solve it in time

$O(n\ d^2)$

Will see this again for probabilistic reasoning!

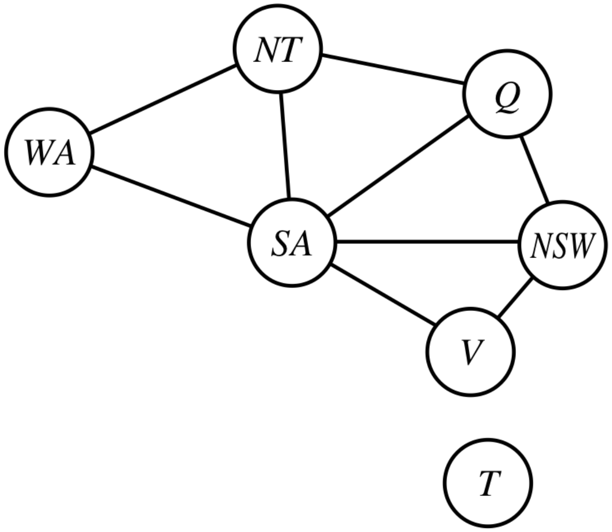
Solving tree structured CSPs

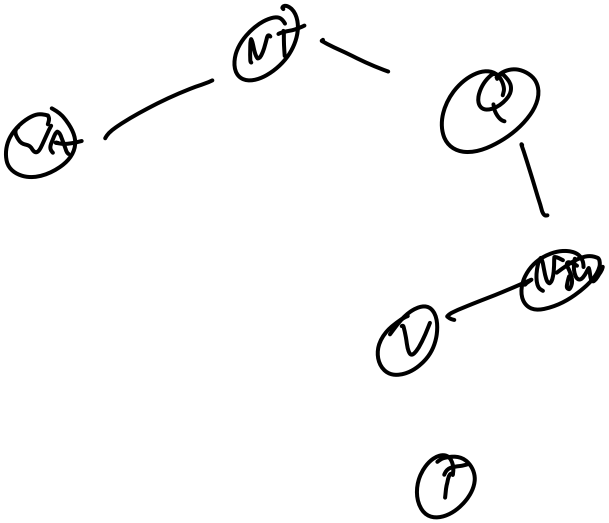


- Choose root; orient edges away from root
- Pick topological ordering
- For j from n down to 1: remove all parent values for which there is no consistent child value
- For j from 1 to n: assign values consistently with parent

- Special case of constraint propagation

Nearly tree-structured CSPs





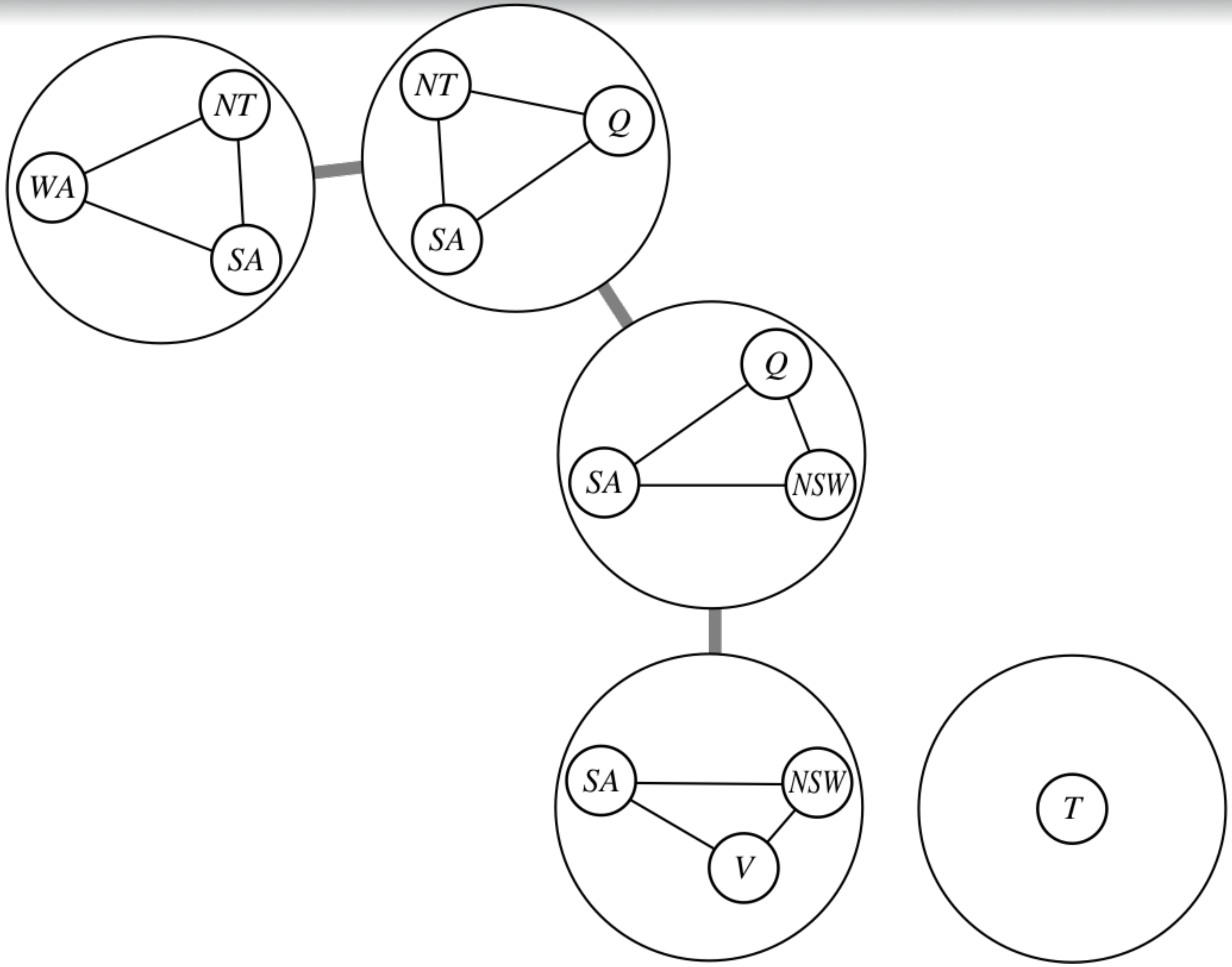
Cutset conditioning

- Pick subset ("cutset") of variables such that remaining variables form a tree
- Search through each possible instantiation of cutset, and try to solve remaining tree-structured CSP

Complexity: Suppose we know the cutset already

$$\underbrace{d^k}_{\text{\# of subproblems}} \cdot \underbrace{(n-k) \cdot d^2}_{\text{solving the tree problem on } n-k}$$

Junction trees (more later)



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  - ➔ Most constrained ➔ Most constraining
- In what order should we try the values?
  - ➔ Least constraining
- Can we detect inevitable failure early?
  - ➔ Forward checking, constraint propagation
- Can we take into account problem structure?
  - ➔ Independent subproblems; trees; tree-like graphs

Summary

- CSPs are special search problem
    - Environment state described using variables
    - Goal test given by constraints
- Backtracking = DFS with fixed var. assigned per node
- Can be sped up using
    - Variable and value selection heuristics
    - Forward checking
    - Constraint propagation / inference
    - Exploit dependency structure among variables