Game Theory: An introduction

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COURSE OUTLINE

Understanding network structure
-- Universal properties of networks
-- What causes these properties?

Exploiting network structure
-- How search works
-- Information cascades & epidemics

Economics and networks
-- Routing & ISP games
-- Ad auctions

Running the web
-- Data center & CDN design
-- MapReduce

Rationality

Suppose that you are perfectly rational. So am I. You know that I am rational. I know you are rational too. You know that I know you are rational. And so on... In this perfectly logical world, how do agents take decisions?

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- All of you value the food items equally: $U(B) = \$9, U(T) = \$8, U(S) = \$5$.
- Costs at Chipotle are as follows: $C(B) = \$7, C(T) = \$5, C(S) = \$4$.

Example contd...

- What would you choose if you just pay for yourself?

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- What would you choose if you just pay for yourself?
- What would you choose if you split the bill evenly?

Example contd...

To each his own:

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- $C(B) = \$7, C(T) = \$5, C(S) = \$4.$

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- $U(B) = \$9, U(T) = \$8, U(S) = \$5.$
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- Oh! Those delicious tacos! Remember you pay \$5.

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- Calculate your own payoffs for each item.
- $$\begin{aligned}\pi(B) &= U(B) - \left(C(B) + c_2 + c_3\right)/3 \\ &= \$(9 - 7/3) - (c_2 + c_3)/3 \\ &= \$(20/3) - (c_2 + c_3)/3.\end{aligned}$$

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- Similarly we get,

$$\begin{aligned}\pi(T) &= \$(19/3) - (c_2 + c_3)/3. \\ \pi(S) &= \$(11/3) - (c_2 + c_3)/3.\end{aligned}$$

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- Aren't those burritos amazing? How much do you pay?
- Wait! Everyone gets a burrito! (Why?) And $7 it is.
- *Rationality hurts!*

Definition

Let us define a common framework to study the effect of rationality!
A game consists of:

- A set of players. Call it $P = \{p_1, p_2, \ldots, p_n\}$.

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- A set of actions for each player.
  Player i's action set $A_i = \{a_i^1, a_i^2, \ldots, a_i^{k_i}\}$.
- Payoffs for each player as a function of the actions taken by all players.
  Payoff for player i is given as $\pi_i(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$.

A Digression

Clarification of terms: *Utility, Cost, Payoff*.
Texts use them in various contexts. Broadly, we have:

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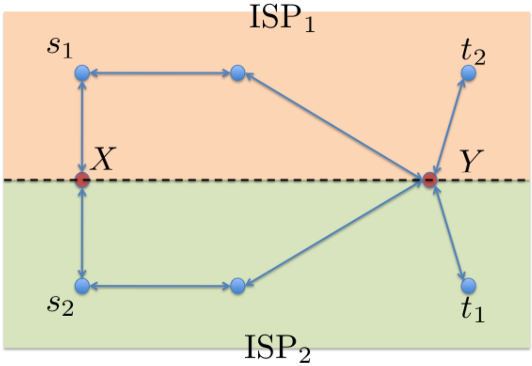
- Utility: A quantitative measure of your satisfaction in consumption.
- Cost: How much you pay for it.
- Payoff: The net benefit measured quantitatively.

Back to the example

- Q: Who are the players?
  A: You and your 2 friends.
- Q: What are their action sets?
  A: Choices of food each player has. Here it is $\{$burrito, tacos, salad bowl$\}$ for all.
- Q: What are the payoffs?
  A: My payoff for a particular set of choices is:
  - *Each his own:*
    $\pi_{\text{me}}(\text{set of choices}) = (\text{utility from my food}) - (\text{cost of my food})$.
  - *Split the bill:* $\pi_{\text{me}}(\text{set of choices}) =$
    $(\text{utility from my food}) - (\text{average cost of food for all})$.

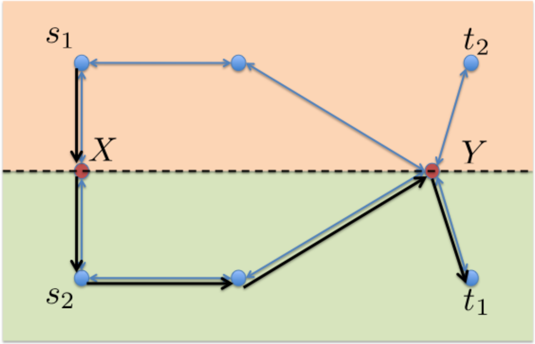
Another example

Consider two ISPs (Internet Service Providers).



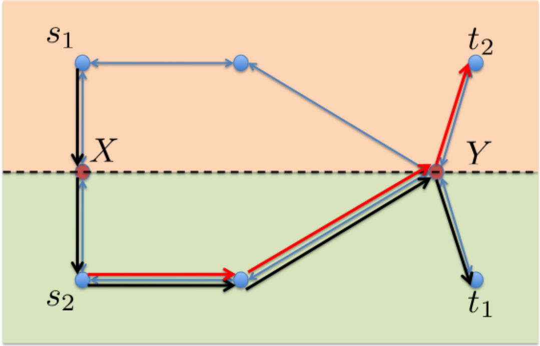
- s_i has traffic for t_i for $i = 1, 2$.
- Cost for each ISP $=$ Sum of number of hops in its own domain for each traffic stream.

Example contd...



- Let A_X denote the action of an ISP using X to pass traffic to the other ISP. Similarly define A_Y.
- Above figure corresponds to ISP_1 playing A_X.

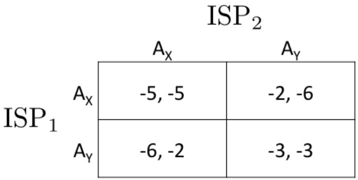
Example contd...



- In this figure, ISP_1 plays A_X, ISP_2 plays A_Y.
- Cost $(\text{ISP}_1) = 2$.
- Cost $(\text{ISP}_2) = 6$.

Example contd...

The *payoff matrix* for this game is given as:



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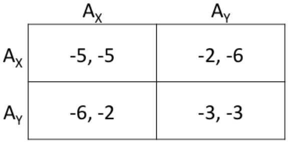

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- Any 2-player one-shot game with finite action sets can be represented as a payoff matrix. This representation is called *normal form* or *strategic form*.

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- (a, b) in the payoff matrix denotes payoffs to players 1 and 2 respectively.

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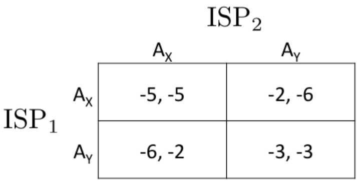

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Consider how ISP$_2$ thinks.

- Q: If ISP$_1$ plays A_X, what should I play?

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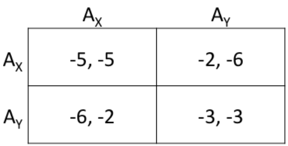

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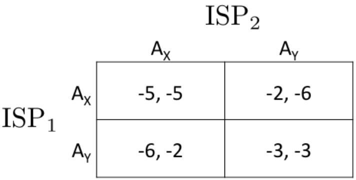

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- Q: If ISP$_1$ plays A_X, what should I play?
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- Q: If ISP$_1$ plays A_Y, what should I play?
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- I should always play A_X!

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Similarly, ISP$_1$ should always play A_X!

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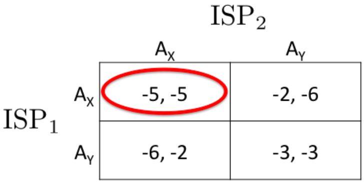

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Thus we end up with this outcome for the game.
Rationality hurts again!

Looking ahead...

Formalizing notions already seen through examples:

- Nash Equilibrium
- Dominant/ dominated strategy.

Looking back...

Remember these lines from the previous slides?

- Q: If ISP_1 plays A_X, what should I play?
- A: I should play A_X. (Why?)
- Q: If ISP_1 plays A_Y, what should I play?
- A: I should play A_X. (Why?)
- I should always play A_X!

Here ISP_2 finds what the *best response* to ISP_1's actions are.

In a general game, a rational player i maximizes his payoff π_i, given others' actions.

Define a_{-i} as actions of other players.

Choose $a_i^* \in A_i$ that solves

$$a_i^*(a_{-i}) = \max_{a_i \in A_i} \pi_i(a_i, a_{-i}).$$

Again, goal of player i is to find the best response to a_{-i}.

Player i's motive

- Player i seeks action a_i that maximizes $\pi_i(a_i, a_{-i})$.

Player *i*'s motive

- Player i seeks action a_i that maximizes $\pi_i(a_i, a_{-i})$.
- Leads us to a natural definition of *equilibrium*, where everyone is playing their best response to others.

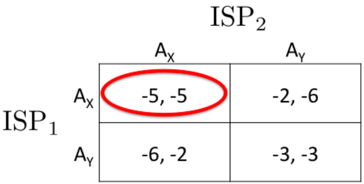
Nash Equilibrium

Definition

A strategy or action profile $(a_1^*, a_2^*, \ldots, a_n^*)$ is a Nash Equilirbium if and only if

$$\pi_i(a_i^*, a_{-i}^*) \geq \pi_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i \quad \forall i$$

Back to ISP Example



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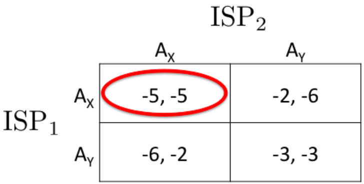

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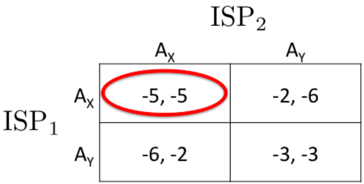

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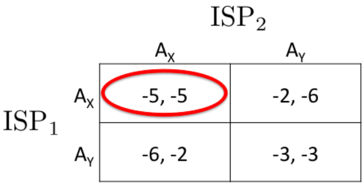

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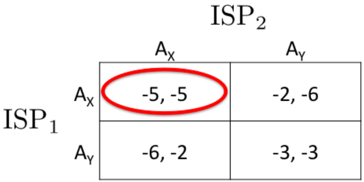

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- Why? Ask yourself the following questions:
- When ISP$_1$ is playing A_X, what should ISP$_2$ play?
- When ISP$_2$ is playing A_X, what should ISP$_1$ play?
- Check with the definition of NE!

A simple example

- Q: Do NE's always exist?

A simple example

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- A: No! Check this example.

P_2

		X	Y
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- Can you make a story for this pay-off matrix?
- This is an example of a *matching pennies* game!

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P_1

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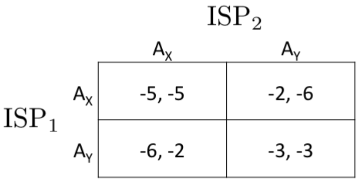
- This is an example of a *coordination* game!

Dominant/ dominated strategy

Consider the ISP example again.



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- Whatever ISP_1 plays, it's best for ISP_2 to play A_X.
- Thus, A_X is a **dominant strategy**. A strategy is *dominant* for a player if he is better off playing it regardless of what the other player chooses.

Dominant/ dominated strategy

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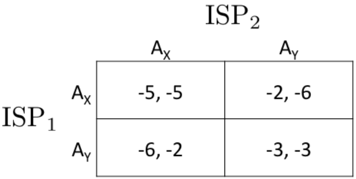

		ISP_2	
		A_X	A_Y
ISP_1	A_X	-5, -5	-2, -6
	A_Y	-6, -2	-3, -3

- Whatever ISP$_1$ plays, it's best for ISP$_2$ to play A_X.
- Thus, A_X is a **dominant strategy**. A strategy is *dominant* for a player if he is better off playing it regardless of what the other player chooses.
- A_Y is a **dominated** strategy. since ISP$_2$ is always better off not playing A_Y.

Dominant/ dominated strategy

Consider the ISP example again.

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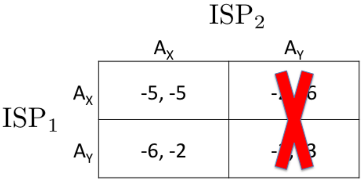
- Thus, ISP$_2$ removes A_Y from its set of strategies.

Dominant strategy contd...

Consider how rationality works.



		ISP_2	
		A_X	A_Y
ISP_1	A_X	-5, -5	-[illegible]6
	A_Y	-6, -2	-[illegible]3

- ISP_1 knows that ISP_2 will remove A_Y from its action set.

Dominant strategy contd...

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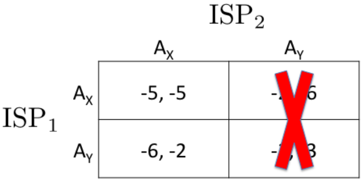

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- ISP$_1$ also removes this column from its strategy space. (Why?)

Dominant strategy contd...

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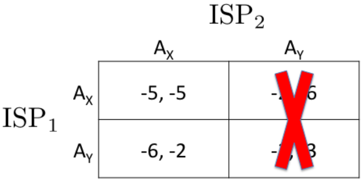

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Dominant strategy contd...

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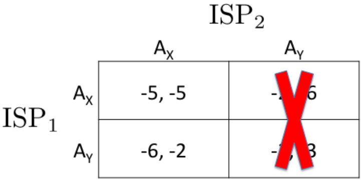

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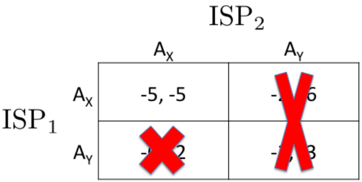
- In this truncated payoff matrix, ISP$_1$ is better off playing A_X.
- And hence...

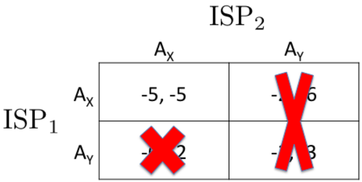
Dominant strategy contd...



		ISP_2	
		A_X	A_Y
ISP_1	A_X	-5, -5	-[illegible]6
	A_Y	-[illegible]2	-[illegible], 3

- We get the Nash Equilibrium of this problem.

Dominant strategy contd...



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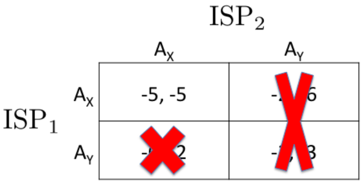

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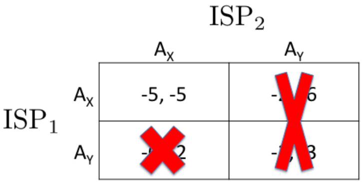

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- We get the Nash Equilibrium of this problem.
- Iteratively remove dominated strategies and get the NE.
- Q: Is it always possible to find NE with this technique?
- A: No! Go back to the simple examples and check.

Recap...

What have we studied so far?

- Definition of a game.

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Moving beyond

Consider the game we've already seen before:

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- NE does not exist here.

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- NE does not exist here.
- Q: Can we do better when we play this game a million times?
- A: Yes! Play probabilistically. Goal is to maximize *expected* payoffs.

Strategy and action

Here we distinguish between an action and a strategy.

- *Action*: It still remains the same.
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- This is called a **mixed strategy**.
- Choosing individual actions is a special case. It's called **pure strategy**.

Mixed Strategy Equilibrium

Define a Nash Equilibrium in this probabilistic setting.

Definition

Suppose $\mathbf{p_1^*}, \mathbf{p_2^*}, \ldots, \mathbf{p_N^*}$ be probability distributions over the action sets $A_1, A_2, \ldots, A_N$ respectively. This set of probability distributions constitutes a **Mixed Strategy Nash Equilibrium** iff

$$\begin{aligned} E[\pi_i(\mathbf{p_1^*}, \ldots, \mathbf{p_i^*}, \ldots, \mathbf{p_N^*})] \\ \geq \quad E[\pi_i(\mathbf{p_1^*}, \ldots, \mathbf{p_i}, \ldots, \mathbf{p_N^*})] \end{aligned}$$

for all distributions p_i over A_i, for all i.

Mixed Strategy Equilibrium

Salient points in the definition:

Explore this definition through an example...

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Mixed Strategy Equilibrium

Salient points in the definition:

- We maximize **expected payoff** here.
- The maximization is over all possible **distributions** over the action sets.
- Each player chooses a best response **distribution**.

Explore this definition through an example...

Example

Consider the game we saw before.

		P_2	
		X	Y
P_1	X	2, 1	1, 2
	Y	1, 2	2, 1

Let's see how player 1 thinks...

Example contd...

- P_1 knows P_2 will randomize between X and Y with probabilities q and $1 - q$.

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- Let's find that.

p as a function of q.

- Let us calculate the expected payoff.

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P_1

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- Expected payoff for P_1:

$$\begin{aligned} E\pi_1(p, q) &= 2pq + 1p(1-q) + \\ &\qquad 1(1-p)q + 2(1-p)(1-q) \\ &= \underbrace{p(2q-1)}_{P_1 \text{ chooses } p} + \underbrace{(2-q)}_{\text{indep. of } p} . \end{aligned}$$

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Goal: Find p that maximizes $E\pi_1(p,q)$:

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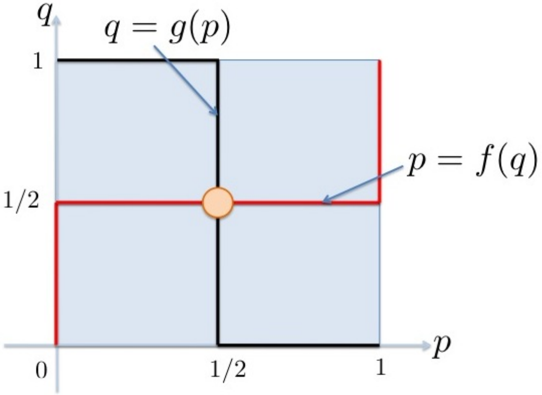
- Similarly find P_2's best response as $q = g(p)$.

Example contd...



- Orange blob is the Mixed Strategy Nash equilibrium. (Why?)
- Again, equilibrium strategies are distributions.

Mixed Strategy NE

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- For a game with finite number of players and finite action sets, it **always exists**.
- The last statement is a loose version of Nash's Theorem.

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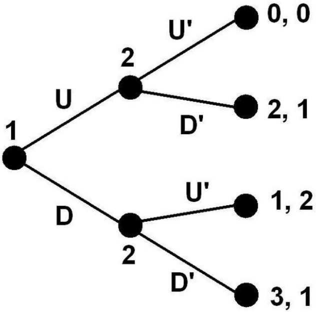
- Extensive form games!
- Use a game tree to capture sequence/timing.

Games in Extensive Form

- Example (perfect information):



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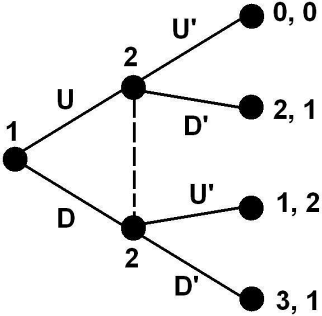
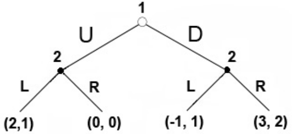


Games in Extensive Form

- Representing extensive form games in strategic/normal form:



a. Extensive Form

	(L,L)	(L,R)	(R,L)	(R,R)
U	2,1	2,1	0,0	0,0
D	-1,1	3,2	-1,1	3,2

b. Strategic Form

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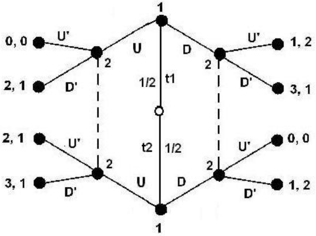

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Later...

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    - Auctions!

Thank you.