CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences
CS 143

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Homework Set #4

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1. Consider a simple queueing system modeled by the differential equation

$$\frac{dx}{dt} = \lambda - \mu, \qquad x \geq 0,$$

where x is the queue length, λ is the arrival rate, and μ is the service rate. Assume that the service rate μ depends on the current length of the queue:

- If there are no capacity restrictions, we take $\mu = x/T$ where T is the time it takes to serve one customer. The service rate thus increases linearly with the queue length.
- As the queue grows longer, we will assume that service rate will be slower because longer queues require more resources, and we assume that the service rate has an upper limit $\mu_{\max}$.

These effects can be captured by modeling the service rate as $\mu = \mu_{\max} f(x)$. The function $f(x)$ is monotone, approximately linear for small x, and $f(\infty) = 1$. For a particular queue, the function can be determined empirically by measuring the queue length for different arrival and service rates. A simple choice is $f(x) = x/(1+x)$, which gives the model

$$\frac{dx}{dt} = \lambda - \mu_{\max}\frac{x}{x+1}. \tag{1}$$

Using equation (1), answer the following questions:

(a) Does the solution to the differential equation (1) exist and is it unique? State any conditions on the parameters λ and $\mu_{\max}$ that might be required for this to be the case. [3 pts]

(b) What are the equilibrium points for the system? Do equilibrium points exist for all possible values of λ and $\mu_{\max}$? [3 pts]

(c) For those values of parameters for which there exists a unique equilibrium point, is the equilibrium point stable? [3 pts]

2. [Low, Exercise 1.3] Suppose two functions f_1 and f_2 are Lipschitz with constants L_1 and L_2, respectively.

(a) Show that the sum $f_{\text{sum}} = f_1 + f_2$ is Lipschitz and provide a Lipschitz constant for f_{sum}. [2 pts]

(b) Show that composition $f_{\text{comp}} = f_1 \circ f_2$ (so $f_{\text{cmp}}(x) = f_1(f_2(x))$) is Lipschitz and provide a Lipschitz constant for f_{comp}. [2 pts]

(c) Provide a counterexample to show that that the product $f_{\text{prod}} = f_1 \cdot f_2$ is not necessarily *globally* Lipschitz. [2 pts]

3. [Low, Exercise 1.4] Consider the following simplified version of the Reno/RED protocol: [8 pts]

$$\dot{x}_i = \left(\frac{1}{T_i^2} - \frac{1}{2} q_i x_i^2\right)_{x_i}^+, \qquad p_\ell = \min\{\rho_\ell b_\ell, 1\},$$
$$q_i = \textstyle\sum_\ell R_{\ell,i} p_\ell,$$
$$\dot{b}_\ell = (y_\ell - c_\ell)_{b_\ell}^+, \qquad y_\ell = \textstyle\sum_i R_{\ell,i} x_i.$$

We can eliminate the prices $p_i(t)$ to obtain an ODE model in terms of $x(t)$ and $b(t)$:

$$\begin{aligned} \dot{x}_i &= \left(\frac{1}{T_i^2} - \frac{1}{2} x_i^2 \sum_\ell R_{\ell,i} \min\{\rho_\ell b_\ell, 1\},\right)_{x_i}^+ =: \Big(f_i(x_i, b)\Big)_{x_i}^+, \\ \dot{b}_\ell &= \left(\sum_i R_{\ell,i} x_i - c_\ell\right)_{b_\ell}^+ =: \Big(g_\ell(x, b_\ell)\Big)_{b_\ell}^+. \end{aligned} \tag{2}$$

Prove that the function (f, g) defined by the right-hand side of equation (2) is Lipschitz on any compact (closed and bounded) set $D \subset \mathbb{R}^{N+L}$.

4. [Low, Exercise 2.1] Prove that the following sets C are convex:

   (a) [affine set] Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ for some $m, n \geq 1$ and define [2pts]

   $$C = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

   (b) [second-order cone] Let $n \geq 1$. For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, introduce the ℓ_2 norm [3pts]

   $$\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

   and define

   $$C = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \leq t\}.$$

5. [Low, Exercise 2.3] Show that the following functions are convex:

   (a) (exponential) $f(x) = e^{ax}$ defined on $C = \mathbb{R}$, where $a \in \mathbb{R}$. [2 pts]

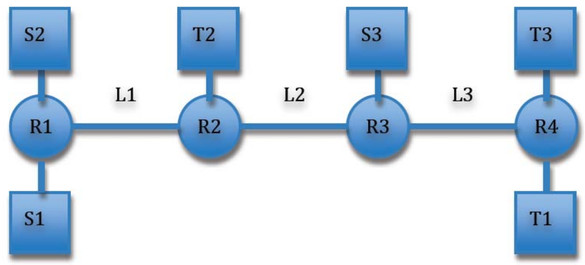
   (b) (logarithm) $f(x) = -\log x$ on $\mathbb{R}_+ = (0, \infty)$. [2 pts]

6. [Low, Exercise 2.5] Show that addition, multiplication by positive constant, and supremum preserve convexity:

   (a) Show that if f_1 and f_2 are two convex functions on the same domain, and $\alpha, \beta \geq 0$, then $\alpha f_1 + \beta f_2$ is also convex. [2 pts]

   (b) Show that if f_1 and f_2 are two convex functions on the same domain, then $f = \max\{f_1, f_2\}$ is also convex. Use this result to show that the function $f(x, y) = |x| + |y|$ defined on $\mathbb{R}^2$ is convex. [2 pts]

7. [Low, Exercise 2.13] Consider the network shown below, where R1–R4 are routers, L1–L3 are links, S1–S3 are source hosts, and T1–T3 are the corresponding destination hosts:



The link capacities of L1, L2 and L3 are 2500 packets/s. The one way propagation delay of each link L1 – L3 is 10ms and assume there is no propagation delay between a host and a router. There are three flows: flow 1 from S1 to T1, flow 2 from S2 to T2, and flow 3 from S3 to T3. Flow 1 starts at $t = 0$ sec, flow 2 starts at $t = 10$ sec, and flow 3 starts at $t = 20$ sec.

(a) Assume that all sources use TCP Reno protocol and that the routers use RED. Using the simplified protocol from Problem 3, calculate the steady-state throughput of each flow and queue length of each link for the periods 0s–10s, 10s–20s, and after 20s. Assume before flow 2 starts, all packets are buffered at L1. [8 pts]

(b) Verify that the equilibrium point that you computed in part 7a is the solution of the network utility maximization problem. (You do not need to independently compute the optimal solution; just confirm that the equilibrium solution satisfies the conditions with the appropriate utility functions.) [3 pts]

(c) Assume that all flows use TCP Vegas (or FAST) for the congestion control protocol, with parameter $\alpha = 50$ and assume each flow "knows" its $\mathrm{RTT}_{\mathrm{min}}$ (round-trip propagation delay) accurately. Recompute the flows from part (a). [8 pts]

(d) Verify that the equilibrium point that you computed in part 7c is the solution of the network utility maximization problem. [3 pts]