# CS/Ec 101 b

1/25/05

# Strategic Form Game

- Players: $i = 1, \ldots, N$
- Strategies: $s^i \in S^i$ for each i
- Payoffs: $u^i(s^1, \ldots, s^N)$ for each i

- Nash Equilibrium:

  $s^* \in S^1 \times \ldots \times S^N$

  $u^i(s^*) \geq u^i(s^*/s^i)$ for each i.

  $(s^*/s^i) = (s^{*1}, \ldots, s^i, \ldots, s^{*N})$

# Next

- An example using Nash Equilibrium
- Existence of Nash Equilibrium
- Why play Nash Equilibrium?
- What is rational?

# Example: Voluntary Contribution Games

- 2 roommates want to buy a home entertainment set. Let y be the size (quality) of the set. A set of size y costs Ky to buy.
- If A pays a and B pays b, then they can get a set of size (a + b)/K.
- A's concave utility from a set of size y is A(y) - a.
- B's concave utility from a set of size y is B(y) - b.
- How big a set do they get? Who pays?

# VC Game

- If B pays b*, then A wants to choose $a \geq 0$ to

  Max $A(a + b^*) - a$

  Or $A'(a + b^*) \leq 1$, where $a > 0$ => = 0.
- Suppose $a^* > 0$

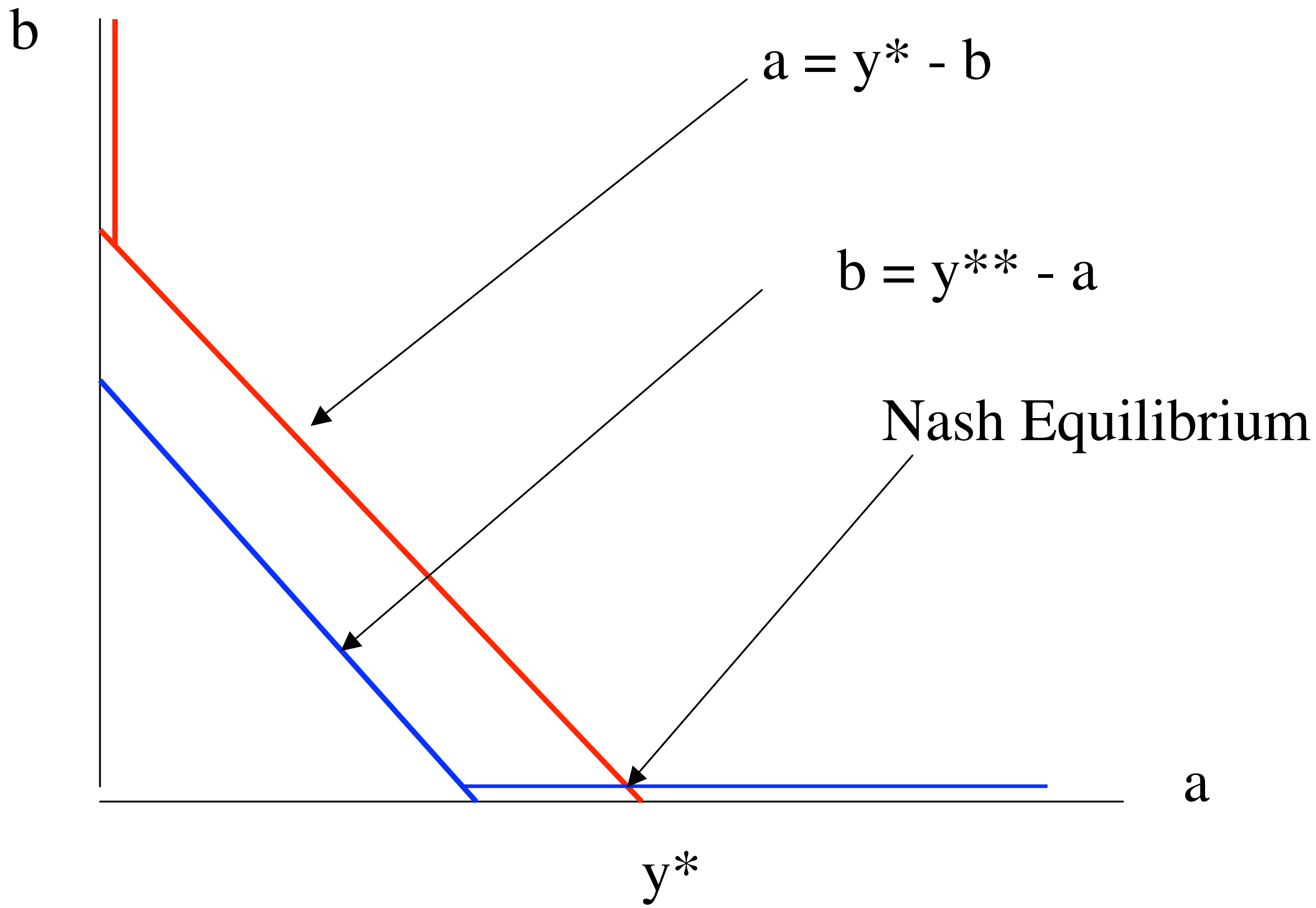
  then $y^* = a^* + b^*$ and $A'(y^*) = 1$.

  So A's best reply function is $a = y^* - b$ if $y^* - b \geq 0$

  Otherwise $a = 0$.
- Similarly for B, but here $B'(y^{**}) = 1$ and

  $b = y^{**} - a$ if $y^{**} - a \geq 0$.

# VC Equilibrium



# Is this a "good" outcome?

- A good outcome is one such that no one can be made better off without making someone worse off. (Pareto-optimal).
- In our example, a good outcome maximizes

  $$A(a+b)+B(a+b) - a - b \text{ with } a, b \geq 0$$

- So $A'(y') + B'(y') - 1 = 0$.

This means the Nash Equilibrium outcome is NOT a good outcome if $B'(y) > 0$, since $y' > y^*$.

# What if one moves first?

- Suppose A moves first.
- Then when B moves, she chooses according to her best replay function.
  - She chooses b = y** - a if ≥ 0.
  - A gets A(y**) - a for all a ≤ y** and
    A(a) - a for all a > y**
  - So A chooses = 0 if A(y**) > A(y*) - y* and
    A chooses y* otherwise.
- Again we do not get the good outcome.

# Let's change the game. Let B subsidize A.

- Suppose B commits to paying A, s for each \$ contributed by A and b = 0.
- Then A gets A(a) - a + sa and

  B gets B(a) - sa.
- Now A'(y") = 1 - s and y" > y*.

  B(y") - sy" - B(y*) > 0 for s near 0.
- So this is good for B.

# The subsidy game (Varian)

- We consider a 2 stage game
  - Stage 1: Simultaneous move to choose s and t, where t is the rate at which A subsidizes B.
  - Stage 2: Simultaneous move to choose contributions a and b.
- What is the sub-game perfect equilibrium?
- If both contribute in equilibrium at stage 2 then

$A'(a+b) - (1-t) = 0$ and $B'(a+b) - (1-s) = 0$.

- If A does not contribute then B could increase t at no cost until A is just on the verge.

# The subsidy game

- So we assume in equilibrium

$A'(a+b) - (1-t) = 0$ and $B'(a+b) - (1-s) = 0$.

- At this point if A were to increase s, then B would contribute everything, $b' = a + b$ and $a = 0$.
- In equilibrium that cannot help A so

$$A(a+b)-(1-t)a - sb \geq A(a+b) - s(a+b)$$

- Similarly a decrease in s can't help A so

$$A(a+b) - (1-t)a - sb \geq A(a+b) - (1-t)(a+b)$$

- Together $b(1-t-s) = 0$. Similarly $a(1-t-s) = 0$.
- So $1 - t - s = 0$.

# The subsidy game

- In equilibrium,

$$A'(a+b) = 1 - t$$

$$B'(a+b) = 1 - s$$

$$1 - s - t = 0.$$

- So $A'(a+b) + B'(a+b) = 1$ and the equilibrium picks a good allocation.

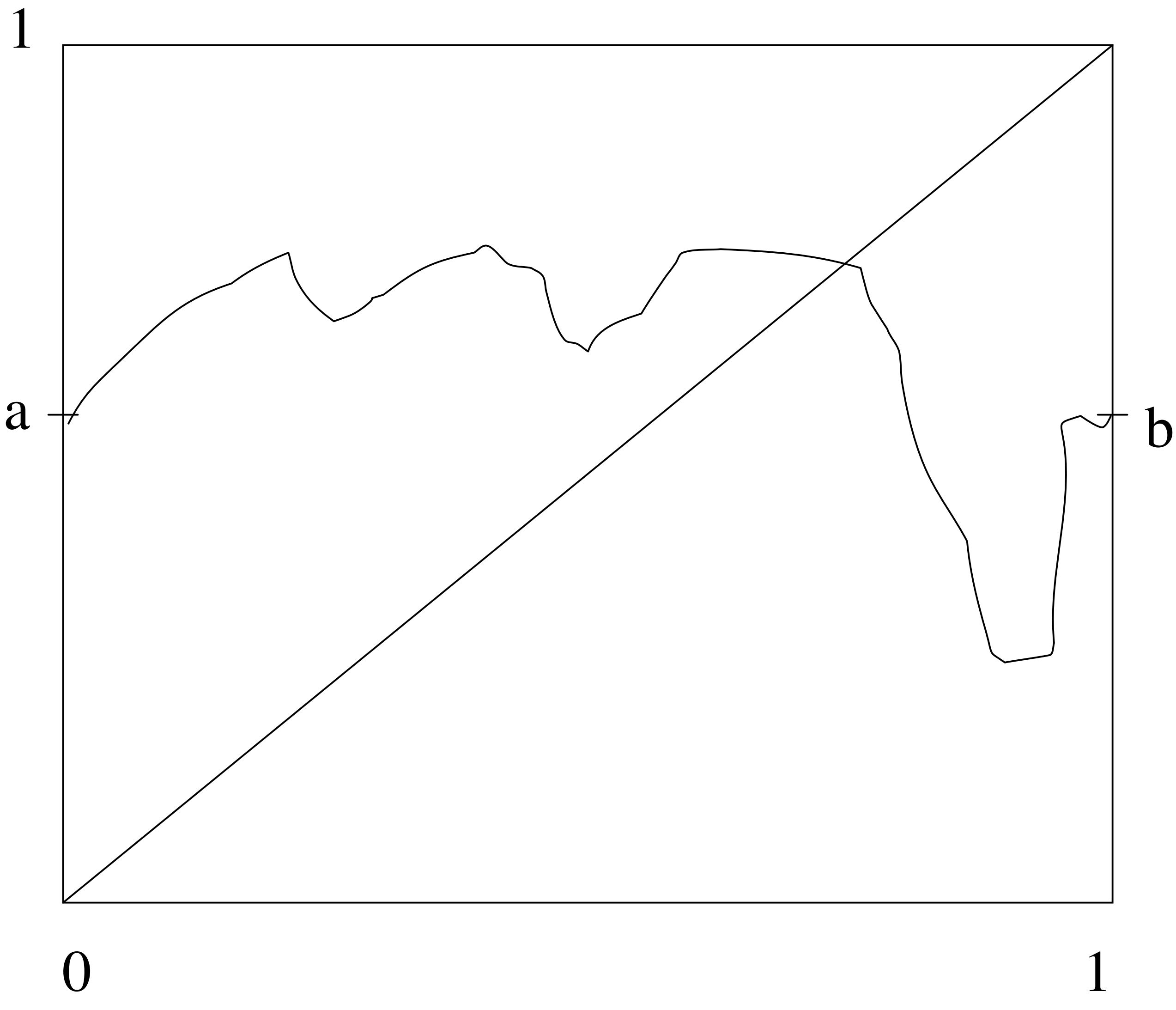
[Note: for those in the know, 1-t and 1-s are the Lindahl equilibrium prices for this situation.]

# More later

- Mechanism design
- Optimal auctions

# When does a Nash Equilibrium Exist?

- The simple answer is “when the reaction functions cross at least once”.
- A mathematical answer relies on a fixed point theorem.
- The vector x is a fixed point of the function f if $x = f(x)$.
- Suppose $f:[0,1] \to [0,1]$ and suppose that $f(0) = a$ and $f(1) = b$. If f is continuous then the curve from a to b must cross the 45 degree line at least once.


1
a
b
0
1

# Fixed point theorem

Correspondence: set-valued function F(x).

Upper semi (hemi) continuous:

- Theorem: (Kakutani)

  X compact, convex.

  F: X => X is a correspondence

  F: upper semi-continuous, convex valued

  Implies

  $\exists$ x* such that x* $\in$ F(x*).

# When does a Nash Equilibrium Exist?

- Given the game $\{I, S^1, \ldots, S^N, u^1, \ldots, u^N\}$ such that

  $S^i$ is compact and convex for all i.

  $U^i$ is continuous in s for all i.

  $U^i$ is concave is $s^i$ for all $s^{-i}$, for all i.

- Let $r^i(t) = \{ s^i \in \text{argmax } u(t/s^i)\}$
- $(r^1(t),\ldots,r^N(t))$ is usc, convex valued.
- There is $t^*$ where $t^* \in r(t^*)$.
- And $t^*$ is a Nash equilibrium.

# Why Play Nash?
# What is Rational?

- We say that player j is <u>rational</u> if they choose a strategy to maximize their payoff given some belief about the other.

# Example: rationality

| | Cooperate | Defect |
|---|---|---|
| Cooperate | 2,2 | 0,3 |
| Defect | 3,0 | 1,1 |

If 2 plays C, then D is best for 1. Similarly if 2 plays D, then D is best for 1.

D is the only rational thing for each. (D, D) is the NE

# But

| | Cooperate | Defect |
|---|---|---|
| Cooperate | 3, 2 | 0, 3 |
| Defect | 2, 0 | 1, 1 |

If 2 plays C, then C is best for 1.
If 2 plays D, then D is best for 1.

What are sensible beliefs for 1 to have?

If 1 believes 2 is rational then 2 will only pick D. So 1 should only believe D, and therefore use D.

Note that (D, D) is the NE.

# Once more…..

| | L | R |
|---|---|---|
| A | 4, 2 | 0, 3 |
| B | 1, 1 | 1, 0 |
| C | 3, 0 | 2, 2 |

A is rational and C is rational.  Why?

L and R are rational. Why?

If 1 assumes 2 is rational then A and C are still ok.

But if 1 assumes 2 assumes 1 is rational then 2 knows 1 will not pick B.  So 2 will pick R.

So 1 picks C.

(C,R) is the NE

# Rationalizable Strategies

- A belief is a probability density $p^i(s^{-i})$ on $S^{-i}$
- We say $s^i$ is rational if there is a $p^i$ such that $s^i$ maximizes $\sum_{s^{-i}} u^i(s^i, s^{-i})\, p^i(s^{-i})$
- We say $s^{*i}$ is rationalizable if for all players j there is a set $Z^j$ such that $s^{*i} \in Z^i$ and for all players $s^{*j}$ and all $s^j \in Z^j$, $s^j$ is a best response to a belief of j such that $p^j(S^{-j}) = 1$.

# Why Play Nash?

- Theorem: Every action used with some probability in some mixed strategy Nash Equilibrium is rationalizable.
- Warning: Other strategies may also be rationalizable.

# Why Play Nash? Domination

- Given the game $\{I, S^1, \ldots, S^N, u^1, \ldots, u^N\}$
- A strategy $t \in S^i$ dominates $s \in S^i$ if $u^i(x/t) > u^i(x/s)$ for all x of the others.
- A rational player will never play a dominated strategy. So we can delete dominated strategies.

# Deletion of dominated strategies

| | L | R |
|---|---|---|
| U | 0, 0 | 5, 10 |
| D | 10, 5 | 20, 20 |

We are left with the NE.

# Deletion of dominated strategies

| | 1 | 2 | 3 |
|---|---|---|---|
| a | 1, 0 | 6, 4 | 0,9 |
| ~~b~~ | ~~2, 1~~ | ~~0, 2~~ | ~~3, 0~~ |
| c | 3, 7 | 2, 3 | 4,0 |

Looks like the end. But a mixed strategy of (1/2, 0, 1/2) dominates strategy 2. Even if 2 is not dominated by a pure Strategy. So delete 2.

# Deletion of dominated strategies

| | 1 | 2 | 3 |
|---|---|---|---|
| a | 1, 0 | 6, 4 | 0,9 |
| b | 2, 1 | 0, 2 | 3, 0 |
| c | 3, 7 | 2, 3 | 4,0 |

Left with the unique Nash Equilibrium.

# Iterated Elimination of Dominated Strategies

- <u>Theorem:</u> There is a unique set, R, of strategy profiles, s, that survives the iterated elimination of dominated strategies.

- <u>Theorem:</u> If the pure strategies of S are finite then the set R is the set of rationalizable strategies.

# What about <u>weakly</u> dominated strategies?

| | l | r |
|---|---|---|
| u | 0, 0 | 0, 1 |
| d | 1, 0 | 0, 0 |

NE are (u,r), (d,r), (d,l)

Sequence of deletions matters: try u then l or r and then try l and then u or d. NE always left but may lose some!