Duality

Nuggets: Remember them!

- Maximize concave function over a a convex set: then …..
- Maximize convex function over a convex set which is also closed and bounded: then……

- Relationship between geometry and algebra: extreme point is what in terms of algebra?

- Simplex algorithm in terms of geometry, algebra and economics.

Nuggets: Remember them!

- Maximize concave function over a a convex set: then **local maximum is a global maximum**
- Maximize convex function over a convex set which is also closed and bounded: then……

there exists an extreme point which is a global maximum

- Relationship between geometry and algebra: extreme point is what in terms of algebra?

Basic feasible solution

- Simplex algorithm in terms of geometry, algebra and economics.

Simplex algorithm

- Geometry:

current_solution is an extreme point;

while (*current_solution* is not a local maximum) {

  find an edge to a higher-valued extreme point;

  current solution = higher-valued extreme point;

}

Simplex Algorithm

- Algebra:

current_solution is basic feasible solution;

while (*current_solution* is not a local maximum) {

find a non-basic variable x_k to increase in value and that increases the objective;

determine which basic variable x_j reduces to zero first as the non-basic variable is increased;

current solution = new basic feasible solution with x_k replaced in the basis by x_j;

}

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

A basic feasible solution (and hence an extreme point) is s = b, x = 0

Is this a local maximum?

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Is this a local maximum?

No, because increasing x_0 by delta (sufficiently small) increases z by 4*delta.

Let us increase x_0. Suppose we keep all other non-basic variables (i.e., x_1 and x_2) at value 0; how large can x_0 become while satisfying constraints?

Example of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
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$$x_0, x_1, x_2, s_0, s_1 >= 0$$

Old basic variables: s_0, s_1.

New basic variables: x_0, s_1.

Convert to canonical form for new basic variables.

Convert to canonical form by pivoting on the element in the column of the incoming basic variable (column 1) and in the row of the outgoing basic variable (row 1).

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$\boxed{1.x_0} + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$

$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

What basic variable drops to 0 first when x_1 is increased?

When x_1 is increased to 4, s_1 drops to 0 while x_0 remains positive. So, the new basic variables are x_0 and x_1.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 0x_1 - 2x_2 - 5s_0 - 2s_1 = z - 24$

Subject to:

$$1.x_0 + 0x_1 + 2/3x_2 + 1/3s_0 - 1/3s_1 = 1$$
$$0.x_0 + 1x_1 + 5/3x_2 - 1/3s_0 + 2/3s_1 = 4$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

Problem is once again in canonical form.

The basic feasible solution for this canonical form is $x_0 = 1$, $x_1 = 4$, with all other variables x_2, s_0, s_1 being 0.

Since all coefficients of c are now negative, **the solution is a local (and hence global) maximum.**

Simplex in terms of Matrices

$$\begin{array}{|c|c|} \hline c_B & c_D \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline B & D \\ \hline \end{array} \begin{array}{|c|} \hline x_B \\ \hline x_D \\ \hline \end{array} = \begin{array}{|c|} \hline b \\ \hline \end{array}$$

Basic solution: $x_B = B^{-1}.b$; $x_D = 0$; $z = c_B.x_B$

Simplex in terms of Matrices

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Basic solution is feasible if and only if: x_B is non-negative (all elements of x_B are non-negative).

Simplex in terms of Matrices

Basic solution: $x_B = B^{-1}.b$; $x_D = 0$; $z = c_B.x_B$

$$\begin{array}{|c|c|}\hline c_B & c_D \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline B & D \\ \hline \end{array} \; \begin{array}{|c|}\hline x_B \\ \hline x_D \\ \hline \end{array} = \begin{array}{|c|}\hline b \\ \hline \end{array}$$

Basic solution is locally (and hence globally) maximum if:
$c_D - c_B.B^{-1}.D$ is non-positive (all elements are non-positive).

Simplex in terms of Matrices

Basic solution: $x_B = B^{-1}.b$; $x_D = 0$; $z = c_B.x_B$

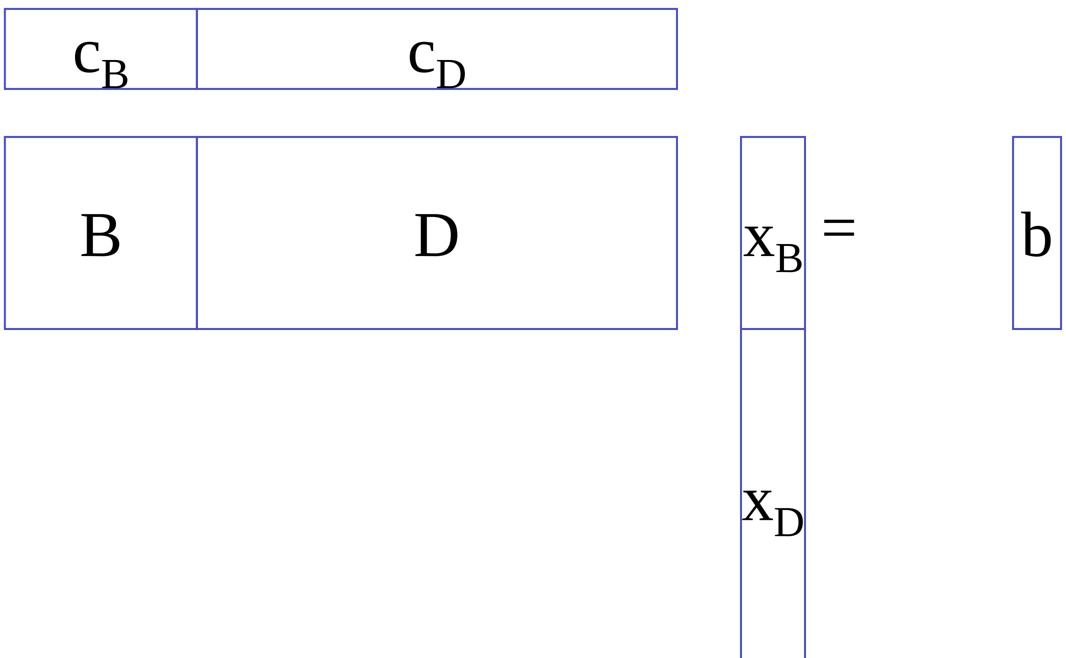
$$\begin{array}{|c|c|}\hline c_B & c_D \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline B & D \\ \hline\end{array} \begin{array}{|c|}\hline x_B \\ \hline x_D \\ \hline\end{array} = \begin{array}{|c|}\hline b \\ \hline\end{array}$$

Let $u = c_B.B^{-1}$. We will see the relationship between u and prices, and between u and dual variables. Describe local max in terms of u.

Simplex in terms of Matrices

Basic solution: $x_B = B^{-1}.b$; $x_D = 0$; $z = c_B.x_B$



Basic solution is global maximum if: $c_D - u.D$ is non-positive.

Basic Variables

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

Suppose basic variables are x_0 and s_1.

What is x_B? x_D? B? D? c_B? c_D? u?

Basic Variables

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Suppose basic variables are x_0 and s_1.

What is $c_D - u.D$?

Is this basic feasible solution a maximum?

Economic Argument for Simplex

- Each constraint specifies the amount of a resource that is available.
- For example, first constraint is amount of machine 1 available to a company and the second constraint is the amount of machine 2 available.
- The variables correspond to different activities: x_0 is the number of chairs made, x_1 the number of tables, x_2 the number of beds, s_0 is discarding machine 1 units, ….
- Making one unit of chairs requires 2 units of machine 1 and 1 unit of machine 2: the column of A corresponding to x_0 . One unit of chairs yields a revenue of 4 units.

Economic Argument for Simplex

- $u = c_B.B_{-1}$
- $z = u.b$
- So, if b_1 is increased by delta, then z increases by u_1.delta.
- So u_1 is the imputed price for the first resource; it is the maximum amount that the company will pay for an increment of that resource (assuming the company's activities are specified by the basic variables x_B .)

Imputed Prices

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
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$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

Suppose basic variables are x_0 and s_1.

u = [2 0].

If the first resource is increased from 6 to 6.001 how much is z increased by (while the only activities carried out correspond to x_0 and s_1)?

If the second resource is increased from 9 to 9.001 what happens?

Imputed Prices

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

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When u = [2 0] does executing activity 2 produce profit where profit = revenue – total price paid for resources?

Imputed Prices

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$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

When u = [2 0] does executing activity 2 produce profit where profit = revenue – total price paid for resources?

Revenue per unit of activity 2 = 9

Resource 1 units consumed per unit of activity = 3

Cost of resource 1 units consumed per unit of activity = 3 * 2

Resource 2 units consumed per unit of activity = 4

Cost of resource 2 units consumed per unit of activity = 0 * 4

Profit = 9 – 6 = 3

Imputed Prices

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

When basic activities are activity 0 and activity 1, what are the prices?

Imputed Prices

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

When basic activities are activity 0 and activity 1, what are the prices?

Prices are [1 2].

Does doing any activity other than the basic activities decrease profit?

Imputed Prices

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
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$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

When basic activities are activity 0 and activity 1, what are the prices?

Prices are [1 2].

Does doing every activity other than the basic activities decrease profit?

Consider activity 2:

Revenue per unit = 9

Cost for resources = 1*3 + 2 *4 = 11

Profit per unit = 9 – 11 = -2.

Yes, all activities reduce profit.

The Primal and Dual Problems

- Primal problem: Max c.x subject to A.x =< b, x >= 0.
- Dual problem: Min u.b subject to u.A >= c, u >= 0.

- The price-based meaning of the dual problem is:

1. u.A >= c means prices are such that no activity k yields a positive profit (c_k – u.A_k).
2. u >= 0 means all prices are non-negative.
3. Objective: find prices to minimize total value of resources (subject to constraint that no activity yields positive profit).

Theorem u.b >= c.x

- Proof?

Theorem u.b >= c.x

- Since x >= 0 and u.A >= c: u.A.x >= c.x
- Since u >= 0 and A.x =< b: u.A.x = < u.b
- From the above two inequalities: u.b >= c.x

Conditions for u.b = c.x?

Conditions for u.b = c.x ?

- Since x >= 0 and u.A >= c: u.A.x >= c.x
- Since u >= 0 and A.x =< b: u.A.x = < u.b
- If u.b = c.x: then u.A.x = c.x and u.A.x = u.b.

- When is u.A.x = c.x?

u.A.x = c.x

Is the same as

(c – u.A).x = 0

Is the same as:

For all k: if $(c - u.A)_k > 0$ then $x_k = 0$

Complementary Slackness

- u.b = c.x if and only if:

  For all k: if $c_k < (u.A)_k$ then $x_k = 0$

  For all k: if $b_k > (A.x)_k$ then $u_k = 0$

- In economic terms

if $c_k < (u.A)_k$ then $x_k = 0$ means: if executing activity k strictly decreases revenue then don't execute activity k.

if $b_k > (A.x)_k$ then $u_k = 0$ means: if a resource isn't scarce then it has no value (no positive price).

Kuhn Tucker Conditions

- The simplex algorithm tells us that the complementary slackness condition can be reached.
- This is generalized by the Kuhn-Tucker conditions that are discussed in other classes at Caltech.