Fundamental Theorems of Optimization

Fundamental Theorems of Math Prog.

- Maximizing a concave function over a convex set.
- Maximizing a convex function over a closed bounded convex set.

Maximizing Concave Functions

- The problem is to maximize a concave function over a convex set.

FUNDAMENTAL THEOREM 1:

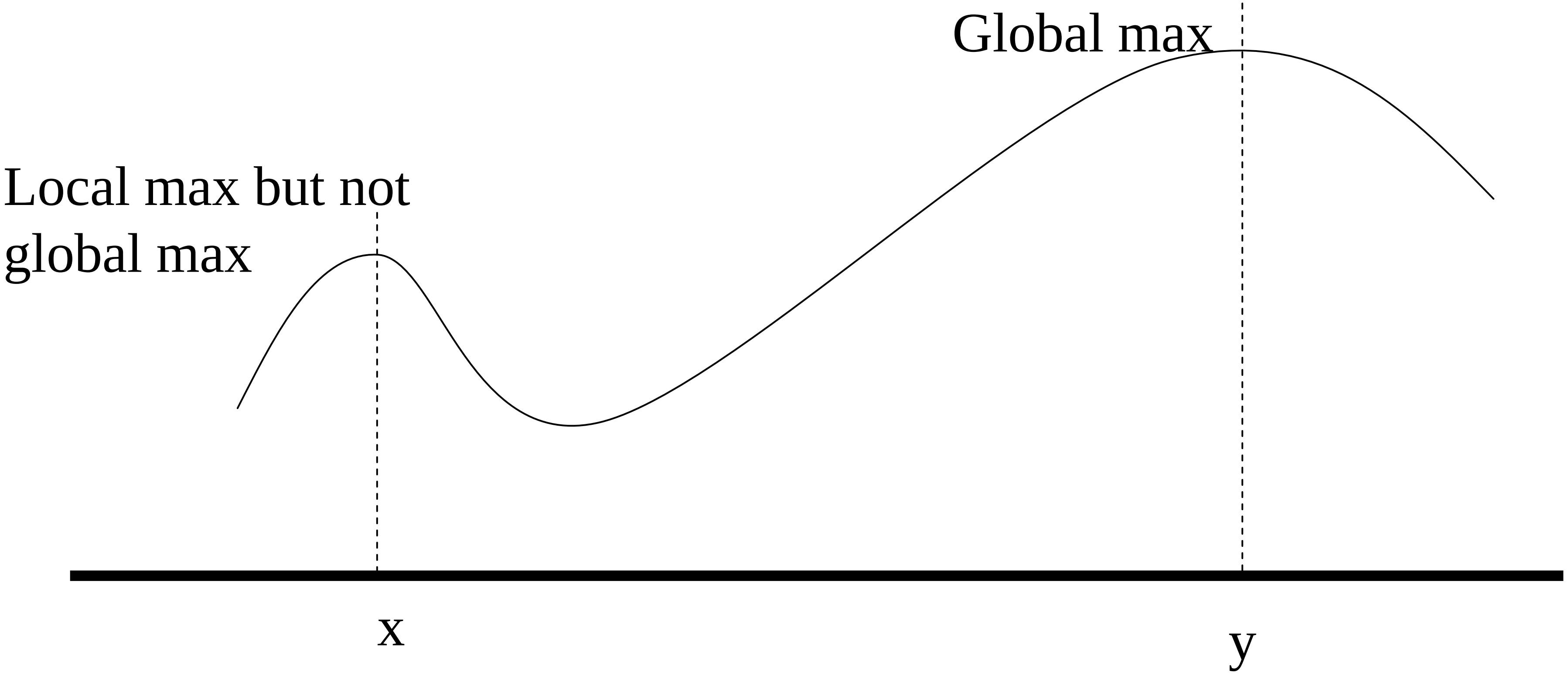
A local optimum is a global optimum.

Proof

- Assume the contrary.
- Let point x be a local optimum that is NOT a global optimum, and let point y be a global optimum.

- Consider the line segment between points x and y.

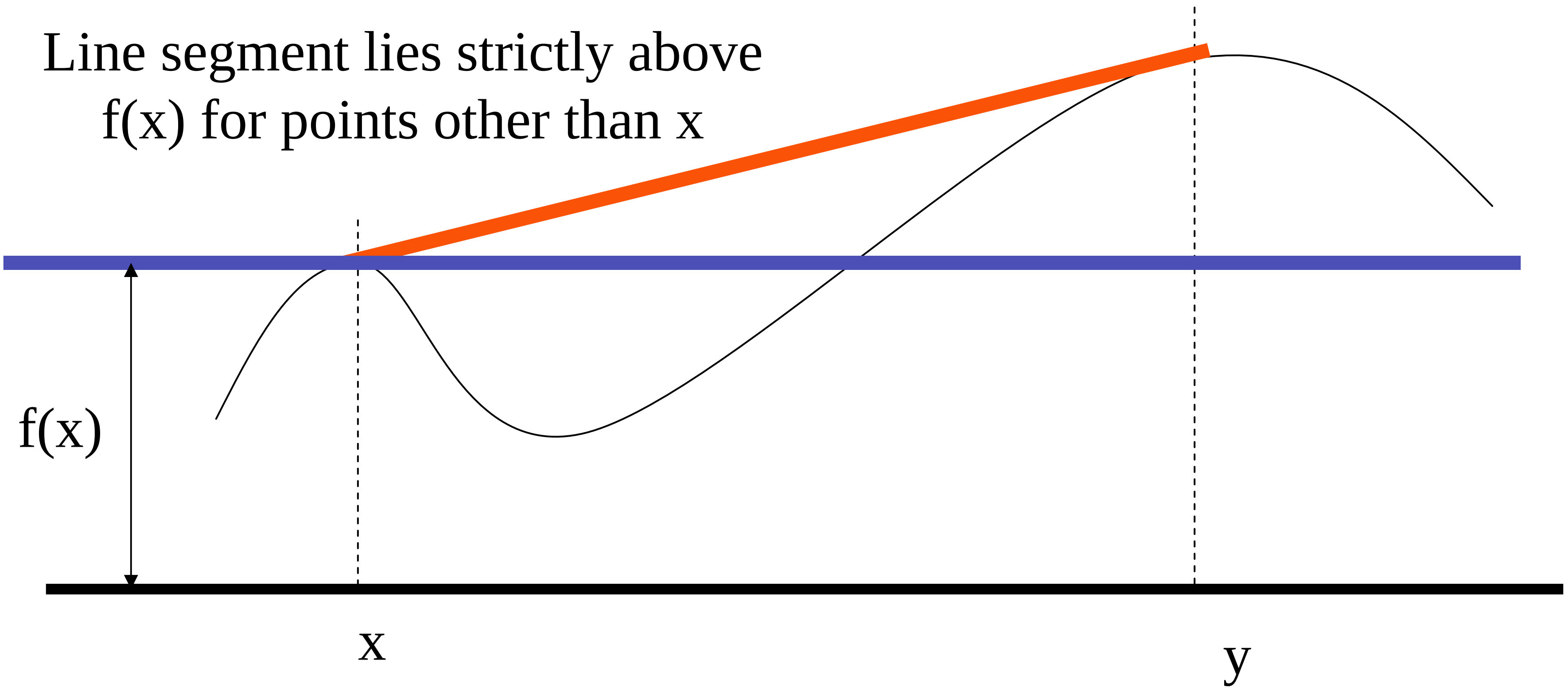
Proof



Proof

- Since f(y) > f(x), the value of f for every point on the line segment between x and y other than x itself, is strictly greater than f(x).

Proof

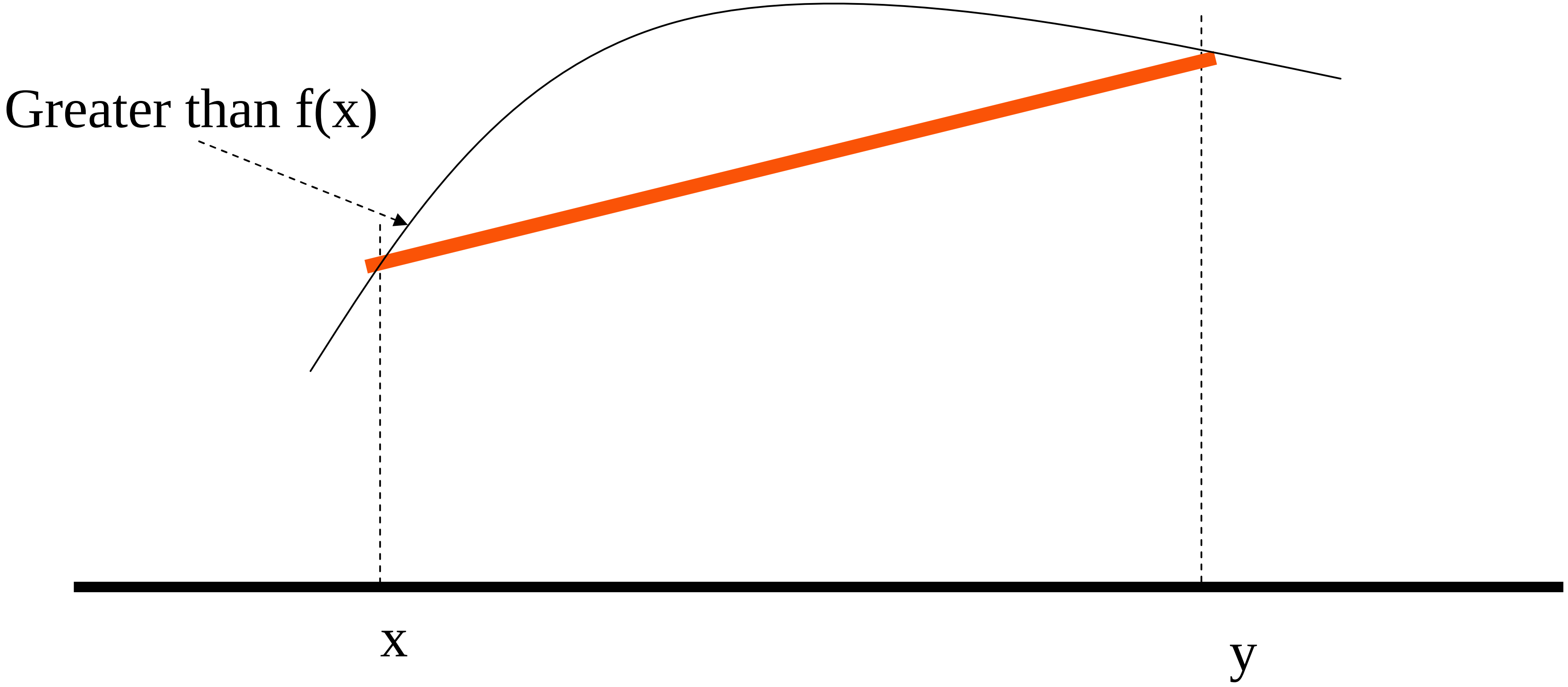

Line segment lies strictly above
f(x) for points other than x
f(x)
x
y

Proof

- Since the curve is concave, the value of f for every point on the line segment is greater than or equal to the value of f on the line segment between points:

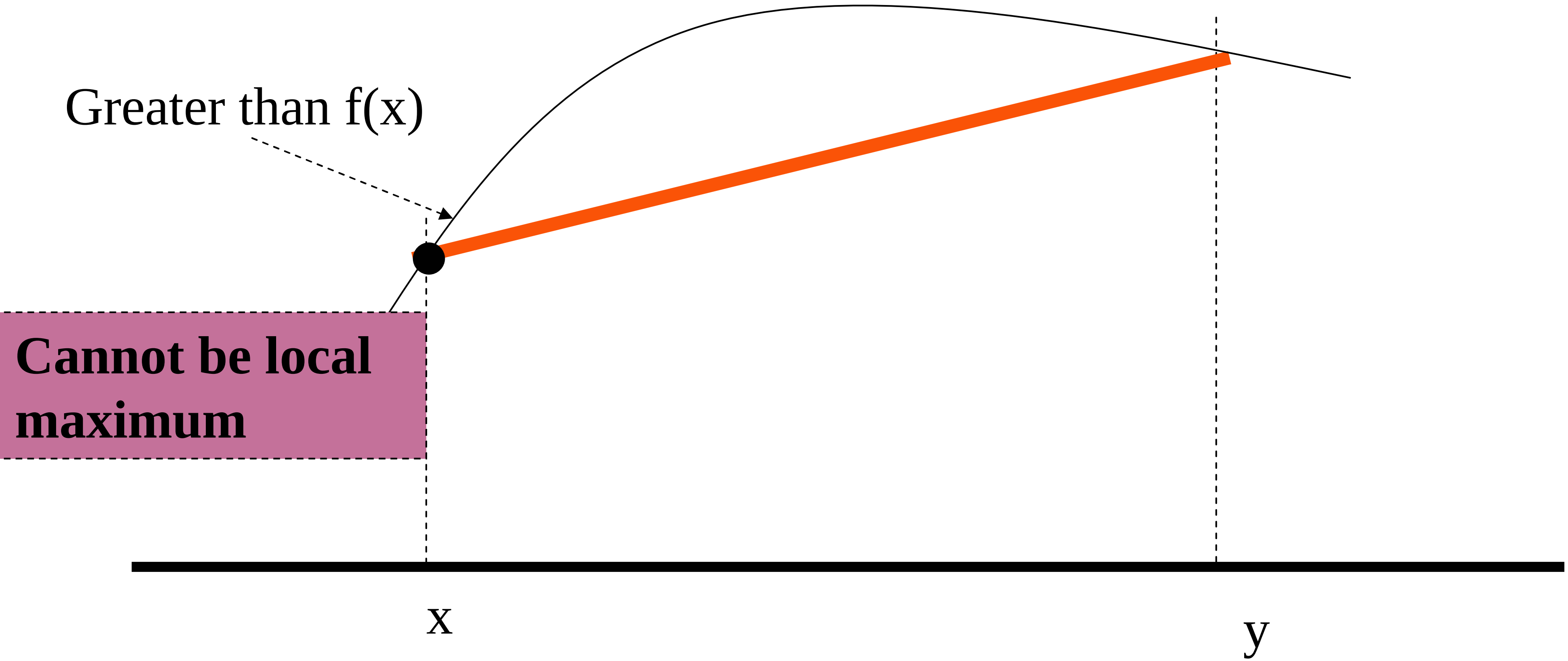
  $$(x, f(x)) \text{ and } (y, f(y))$$

Proof



Curve lies strictly above
the line segment for points other than x

Proof by Contradiction

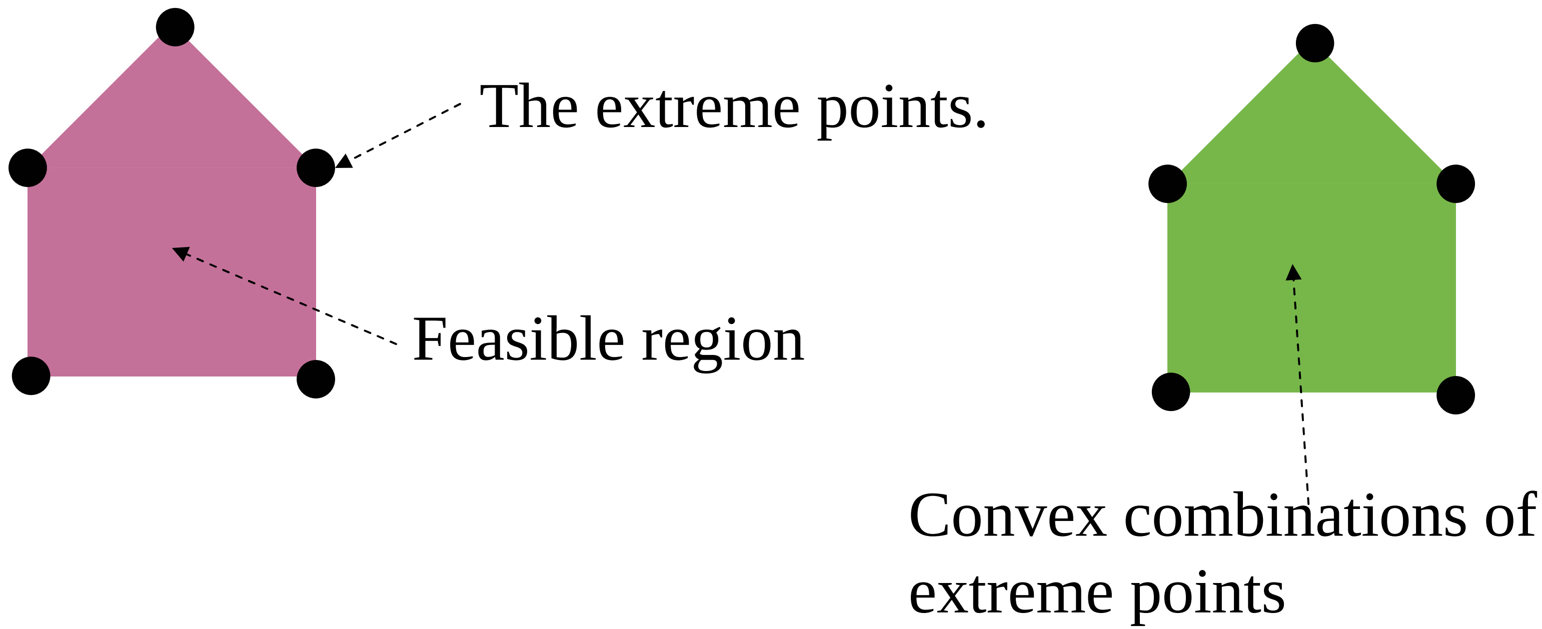


Curve lies strictly above
the line segment for points other than x

Maximizing Convex Functions

- Problem: Maximizing convex functions over closed bounded convex sets.
- Because the feasible region is a closed bounded convex set, the feasible region is the set of points that are convex combinations of extreme points.

Feasible Region and Extreme Points



Feasible region is the set of all convex combinations of extreme points

Maximizing Convex Functions

Fundamental Theorem 2

There exists an extreme point which is the global maximum of a convex function over a closed bounded convex set.

Proof by Contradiction

- Assume the contrary.
- So there is a global maximum at a point p that is not an extreme point, and no global maximum occurs at an extreme point.
- Therefore for every extreme point q:

$$f(p) > f(q)$$

- Point p is a convex combination of some set of extreme points. So there exists extreme points $x_1, x_2, \ldots, x_k$ and positive scalars $r_1, \ldots, r_k$, that sum to 1 such that:

$$p = \text{sum over } j \text{ of } r_j * x_j$$

Proof

- Since f is a convex function:

  f(sum over j of r_j * x_j)

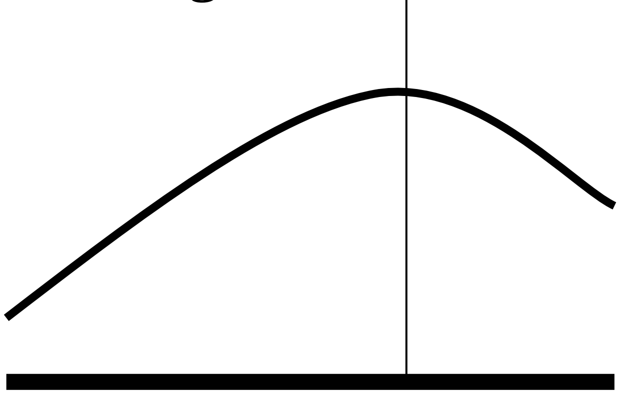
  =<

  Sum over j of r_j * f(x_j)

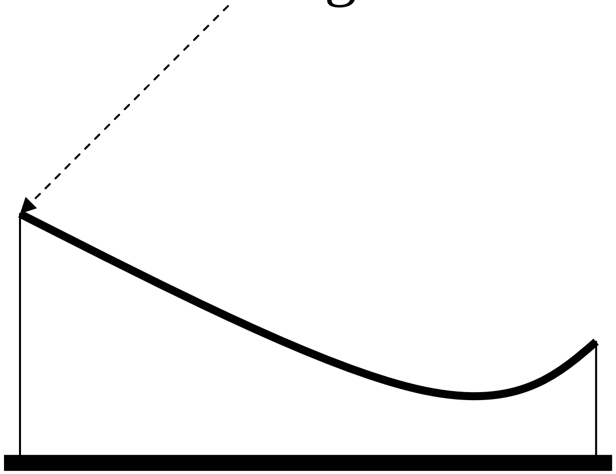
- If for every j, f(x_j) is strictly less than the global maximum, then the right hand side of the above inequality is strictly less than the global maximum, and so the left hand side is also strictly less than the global maximum. But, the left hand side is f(p). Hence p is not a global maximum. Contradiction!

Remembering the Fundamental Theorems Graphically

Local maximum
is
global maximum



There exists an extreme point
which is a global maximum



Introduction to Linear Programming

Maximizing Linear Functions

- Which of the fundamental theorems can we apply if the objective function is linear?

Maximizing Linear Functions

- Which of the fundamental theorems can we apply if the objective function is linear?
- BOTH fundamental theorems because linear functions are both concave functions and convex functions!

Maximizing Linear Functions

- Maximizing linear functions over closed bounded convex sets allows us to use both theorems. So:
  - **Every local maximum is a global maximum, and**
  - **There exists an extreme point that is a global maximum.**

Algorithm for Linear Objective Functions

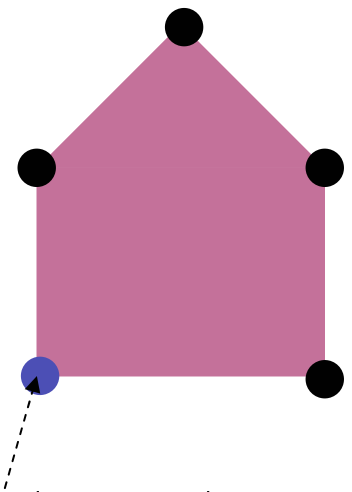
- Start at an extreme point.
- While the extreme point is not a local maximum **do**

traverse along the boundary of the feasible region to a better neighboring extreme point, i.e., a point with a higher value of the objective function.

Linear Feasible Regions

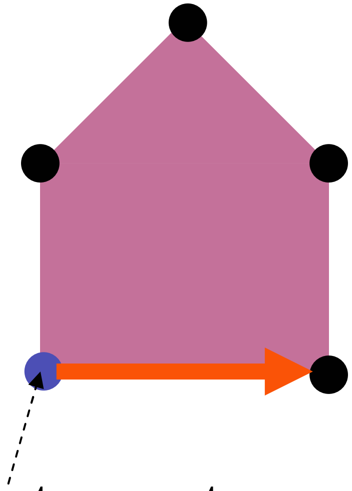
- Can we simplify the problem if the feasible region is bounded by hyperplanes?
- The feasible region is a polyhedron with a finite number of extreme points.
- So, we know the algorithm will stop because the while loop never returns to the same extreme point.
- So, the loop can iterate at most N times where N is the number of extreme points of the polyhedron representing the feasible region.

Algorithm for Linear Problems



Start at an extreme point.
If it is not a local maximum find a direction of improvement along a boundary to a neighboring better extreme point.

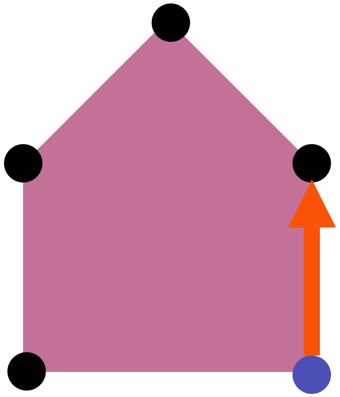
Algorithm for Linear Problems



Start at an extreme point.

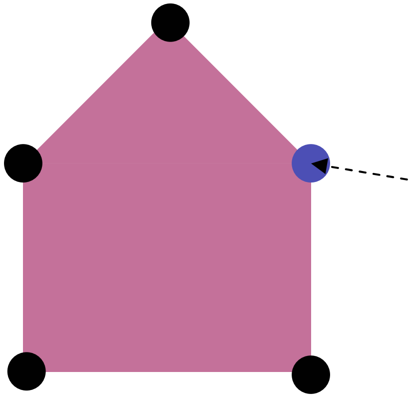
If it is not a local maximum find a direction of improvement along a boundary to a neighboring better extreme point.

Algorithm for Linear Problems



If the extreme point is not a local maximum find a direction of improvement along a boundary to a neighboring better extreme point.

Algorithm for Linear Problems



If the extreme point is a local maximum, stop.

Linear Programming

Linear programming is mathematical programming where the objective function and constraints are linear.

Example:

Maximize z

Where $3x_0 + 2x_1 = z$

Subject to:

$$2x_0 + x_1 =< 4$$

$$x_0 + 2x_1 =< 6$$

$$x_0, x_1 >= 0$$

Canonical Form

For now, we restrict ourselves to problems of the form:

Maximize z where

c.x = z

Subject to:

A.x =< b

x >= 0

Where:

- c is an row vector of length n,
- x is a column vector of length n,
- A is an m x n matrix, and
- b is a column vector of length m with all values non-negative

Example of Canonical Form

Example:
Maximize z
Where $3x_0 + 2x_1 = z$
Subject to:

$$2x_0 + x_1 =< 4$$
$$x_0 + 2x_1 =< 6$$

$$x_0, x_1 >= 0$$

$C = [3, 2]$

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Another Canonical Form

Convert the inequalities to equalities.

Example:
Maximize z
Where $3x_0 + 2x_1 = z$
Subject to:

$$2x_0 + x_1 =< 4$$
$$x_0 + 2x_1 =< 6$$

$$x_0, x_1 >= 0$$

Example:
Maximize z
Where $3x_0 + 2x_1 + 0s_0 + 0s_1 = z$
Subject to:

$$2x_0 + x_1 + s_0 + 0s_1 = 4$$
$$x_0 + 2x_1 + 0s_0 + s_1 = 6$$

$$x_0, x_1, s_0, s_1 >= 0$$

The Alternative Canonical Form

Max z
Where $c.x = z$

Subject to:
A. $x =< b$
$x >= 0$

Max z
Where $c.x + 0.s = z$

Subject to:
A. $x + I.s = b$
$x, s >= 0$

Here I is the identity matrix

Relationship to Economics

- The constraints represent constraints on resources.
- The columns represent activities.
- The objective function represents revenue.

Relationship to Economics

Maximize z

Where $4x_0 + 5x_1 + 9x_2 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 =< 6$$

$$x_0 + 2x_1 + 4x_2 =< 9$$

$$x_0, x_1 >= 0$$

Example: A furniture maker has two scarce resources: wood and labor. The company has 6 units of wood and 9 units of labor. The company can make small tables, chairs, or cupboards. A table requires 2 units of wood and 1 unit of labor and produces revenue of 4 units. A chair requires 1 unit of wood and 2 units of labor and produces a revenue of 5 units. A cupboards requires 3 units of wood, 4 of labor and produces a revenue of 9 units. How many tables, chairs and cupboards should the company make to maximize revenue?

Definition: Basic Feasible Solution

- A feasible solution is a vector X such that:

  A.X =< b, and X >= 0.

- A basic solution is one in which at most m variables are non-zero (and so at least n-m variables are strictly zero), and the m, possibly non-zero variables correspond to linearly independent columns.
- Let B be the matrix obtained by putting together the columns of the possibly non-zero variables. Let the m-vector formed by putting these variables together be x_B. Then the basic solution is

  $$x_B = B^{-1}.b,$$

  and all other variables are strictly zero.

Examples of Basic Solutions

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$

$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1 >= 0$$

Example 1:

Basic variables are s_0, s_1.

Non-basic variables are x_0, x_1 , x_2.

The columns corresponding to the basic variables are the last two columns, i.e.,

1 0

0 1

So, the basic solution is

s_0, s_1 = 6, 9 and x_0, x_1 , x_2 = 0, 0, 0

Examples of Basic Solutions

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1 >= 0$$

Example 2:

Basic variables are x_0, x_1.

Non-basic variables are x_2, s_0, s_1.

The columns corresponding to the basic variables are the first two columns, i.e.,

2 1
1 2

So, the basic solution is

$x_0, x_1 = 1, 4$ and $x_2, s_0, s_1 = 0, 0, 0$

Examples of Basic Solutions

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1 >= 0$$

Example 3:

Basic variables are x_1, s_0.

Non-basic variables are x_0, x_2 , s_1.

The columns corresponding to the basic variables are the first two columns, i.e.,

1 1

2 0

So, the basic solution is

$x_1, s_0 = 4.5, 1.5$ and

$x_0, x_2 , s_1 = 0, 0, 0$

Economic Meaning of Positive Slack Variables

- The slack variable is the variable we added to turn a constraint into an equality.
- We have one slack variable for every constraint.
- If a slack variable is positive in a solution, that means that the inequality constraint is not tight. In other words, we are not using all of that resource. This implies that the resource is not scarce.

Theorem

- A solution is an extreme point of the feasible region if and only if it is a basic feasible solution.
- Proof:
- First prove that every basic feasible solution is an extreme point.
- Then prove that every feasible solution that is not a basic solution, is not an extreme point.

Every basic feasible solution is extreme

- Proof:
- Let p be a basic feasible solution. Let u and v be two feasible solutions, distinct from p, such that the line segment between u and v passes through p.
- Thus p is a weighted average of u and v with positive weights.
- Since u and v are feasible, their elements are non-negative.
- The only way that weighted average (with positive weights) of non-negative numbers is strictly zero is for the numbers themselves to be strictly zero too.

Proof continued

- Hence the values of non-basic variables in u and v are strictly zero.
- Therefore, the values of the basic variables in both u and v must be: $B^{-1}.b$
- This is the same as the solution for p.
- Hence u and v are not distinct from p: contradiction!

Theorem: Non-basic is not Extreme

- Proof:
- Assume there are m+1 or more variables that are strictly positive in a solution p.
- The column corresponding to at least one of these variables, say , is linearly dependent on the remaining columns.
- Consider a solution u obtained by perturbing solution p by increasing x_k by arbitrarily small positive epsilon and adjusting m positive variables in p so that u is feasible.
- Consider a solution v, obtained in the same way by decreasing x_k by arbitrarily small positive delta.
- Show that p can be obtained by a linear combination of u and v.

Theorem

- Consider the problem:

Max z where c.x = z subject to A.x + I.s = b, and x,s>=0.

- The basic feasible solution x = 0, s = b is locally optimum if all the elements of c are non-positive.
- Proof: c.x is non-positive since c is non-positive and x is non-negative. Hence z = 0 is an optimal solution.

The Algorithm

- Start with a basic feasible solution: say x = 0, s = b, where s is the vector of slacks.
- While locally-optimum extreme point is not found: do:
  - Increase the value of a non-basic variable that improves the objective function until a basic variable becomes zero.
  - Modify the basis as follows: Replace the basic variable that has become zero by the variable that became positive.

The Algorithm: Computational Steps

- Start with a basic feasible solution: say x = 0, s = b, where s is the vector of slacks.
- Always maintain a problem in the canonical form: max z where c.x + 0.s = z, subject to A.x + I.s = b, and x, s >= 0
- While an element of c is positive: do:
  - Increase the value of a non-basic variable corresponding to a positive c until a basic variable becomes zero.
  - Modify the basis as follows: Replace the basic variable that has become zero by the variable corresponding to the positive element of c.
  - Convert to canonical form.

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

Problem is in canonical form:

Max z

Where c.x + 0.s = z

Subject to:

A.x + I.s = b

x, s >= 0

And where b is non-negative.

A basic feasible solution (and hence an extreme point) is s = b, x = 0

There are coefficients of c that are positive. Increase any variable with a positive coefficient, say x_0

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

Keep increasing x_0 until some currently basic variable decreases in value to 0.

How large can we make x_0?

Which basic variable decreases to 0 first?

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

Keep increasing x_0 until some currently basic variable decreases in value to 0.

How large can we make x_0?

Which basic variable decreases to 0 first?

When x_0 increases to 3, s_0 decreases to 0 while s_1 is still positive. So, at the next basic feasible solution (and hence extreme point) the basic variables are x_0 and s_1, while the other variables, x_1, x_2 and s_0 are strictly 0.

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

Old basic variables: s_0, s_1.

New basic variables: x_0, s_1.

Convert to canonical form for new basic variables.

Convert to canonical form by pivoting on the element in the column of the incoming basic variable (column 1) and in the row of the outgoing basic variable (row 1).

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

pivot ($2x_0$)

Old basic variables: s_0, s_1.

New basic variables: x_0, s_1.

Convert to canonical form for new basic variables.

Convert to canonical form by pivoting on the element in the column of the incoming basic variable (column 1) and in the row of the outgoing basic variable (row 1).

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$2x_0 + x_1 + 3x_2 + s_0 + 0s_1 = 6$$
$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

pivot

To pivot, our goal is to make the column vector into a unit vector. So we want to transform the column [4, 2, 1] to [0, 1, 0]. So, divide first constraint by 2.

Examples of Simplex Algorithm

Maximize z

Where $4x_0 + 5x_1 + 9x_2 + 0s_0 + 0s_1 = z$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$

$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

pivot

To pivot, our goal is to make the column vector into a unit vector. So we want to transform the column [4, 1, 1] to [0, 1, 0]. So, subtract 4 times first constraint from objective function.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$

$$x_0 + 2x_1 + 4x_2 + 0s_0 + s_1 = 9$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

To pivot, our goal is to make the column vector into a unit vector. So we want to transform the column [0, 1, 1] to [0, 1, 0]. So, subtract first constraint from second.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$

$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$

$x_0, x_1, x_2, s_0, s_1 >= 0$

pivot

To pivot, our goal is to make the column vector into a unit vector. So we want to transform the column [0, 1, 1] to [0, 1, 0]. So, subtract first constraint from second.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$
$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

The problem is again in canonical form.

A basic feasible solution is $x_0 = 3$ and $s_1 = 6$

Are there elements of c that are positive?

Yes, coefficient of x_1 is positive. So, increase x_1.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$

$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

What basic variable drops to 0 first when x_1 is increased?

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$
$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

What basic variable drops to 0 first when x_1 is increased?

When x_1 is increased to 4, s_1 drops to 0 while x_0 remains positive. So, the new basic variables are x_0 and x_1.

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$
$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

Old basic variables:
x_0, s_1

New basic variables:
x_0, x_1

Convert to canonical form with new basic variables.

Pivot on column of incoming basic variable and row of outgoing basic variables

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 3x_1 + 3x_2 - 2s_0 + 0s_1 = z - 12$

Subject to:

$$1.x_0 + 0.5x_1 + 1.5x_2 + 0.5s_0 + 0s_1 = 3$$

$$0.x_0 + 1.5x_1 + 2.5x_2 - 0.5s_0 + s_1 = 6$$

$$x_0, x_1, x_2, s_0, s_1 >= 0$$

pivot

Pivot on column of incoming basic variable and row of outgoing basic variables

Examples of Simplex Algorithm

Maximize z

Where $0x_0 + 0x_1 - 2x_2 – 5s_0 - 2s_1 = z - 24$

Subject to:

$$1.x_0 + 0x_1 + 2/3x_2 + 1/3s_0 - 1/3s_1 = 1$$

$$0.x_0 + 1x_1 + 5/3x_2 - 1/3s_0 + 2/3s_1 = 4$$

$$x_0 , x_1 , x_2 , s_0 , s_1 >= 0$$

Problem is once again in canonical form.

The basic feasible solution for this canonical form is $x_0 = 1$, $x_1 = 4$, with all other variables x_2, s_0, s_1 being 0.

Since all coefficients of c are now negative, **the solution is a local (and hence global) maximum.**