CS101, Ec 101

- Special topics in Computer Science.
- Selected topics in Economics

- Instructors:
  - John Ledyard, Economics
  - Mani Chandy, Computer Science
  - Guest lecturers

- Time/place : Tuesday, Thursday 9 – 10:30, Jrg 74

Course Goals

- Survey theories in economics and information science and technology (IST).
- Apply theories to problems at the intersection of economics and IST.

- Emphasis is on:
  - Survey; not in-depth detailed discussion of selected areas
  - Fundamental theorems
  - Pointers to further courses and research opportunities

Course Grading and Load

- No final, mid-terms, or quizzes.
- 6 - 8 homework assignments.
- Possibly a couple of programming assignments.

- The units for this class are fair:
  - 9 units so 9 hours/week for an average Caltech student.
  - 3 hours lecture, 6 hours homework or lab.

Course Support Material

- No text.
- Slides and other notes will be provided in some (but not all lectures). You will be expected to attend lectures and take notes.
- We will cover only the fundamental theorems in class; you are expected to study the rest of the slides and notes on your own.

Research Opportunities

- We will focus on fundamental theorems, and not applications or research opportunities in the lectures (because we don't have time to do both).
- You are encouraged to explore research opportunities.
- Research focuses on integrating economics and IST by studying economic networks with significant physical constraints.
  - Electrical power utility
  - Event Web: Event services utility
  - Gas utility

Research Opportunities

- Electric Utility
  - Dr. Dan Zimmerman, CS
  - Shaun Lee, CS
  - Richard Murray, John Ledyard, Mani Chandy
  - www.surf.caltech.edu/opportunities/abstracts/05EAS_Chandy3.htm
- Event Utility
  - Lu Tian: Economics of events
  - Jonathan Lurie, building the Event Web:
  - www.surf.caltech.edu/opportunities/abstracts/05EAS_Chandy1.htm

Topics (Tentative)

- Optimization: January 4, 6, 11, 13: Mani
- Game Theory: January 18, 20, 25, 27: John
- Decision Theory: February 1, 3, 8, 10: Mani, John
- Computer Science Fundamentals: Economics applications: February 15, 17, 22, 24: Mani
- Fundamentals of Economics; IST applications: March 1, 3, 8

Mathematical Programming: Fundamentals

Outline

- The problem domain
- Definitions

Problem Domain: Optimization

- Maximize f(x)
- Subject to g(x) =< c
  - Where f(x) is a function from a vector space to the reals
  - And g(x) is a function from the vector space to a vector of length m and
  - c is a constant vector of length m.

- f(x) is called the ***objective function***
- g(x) = < c is called the set of ***constraints***.

Problem Domain: Optimization

- Maximize f(x)
- Subject to g(x) = < c

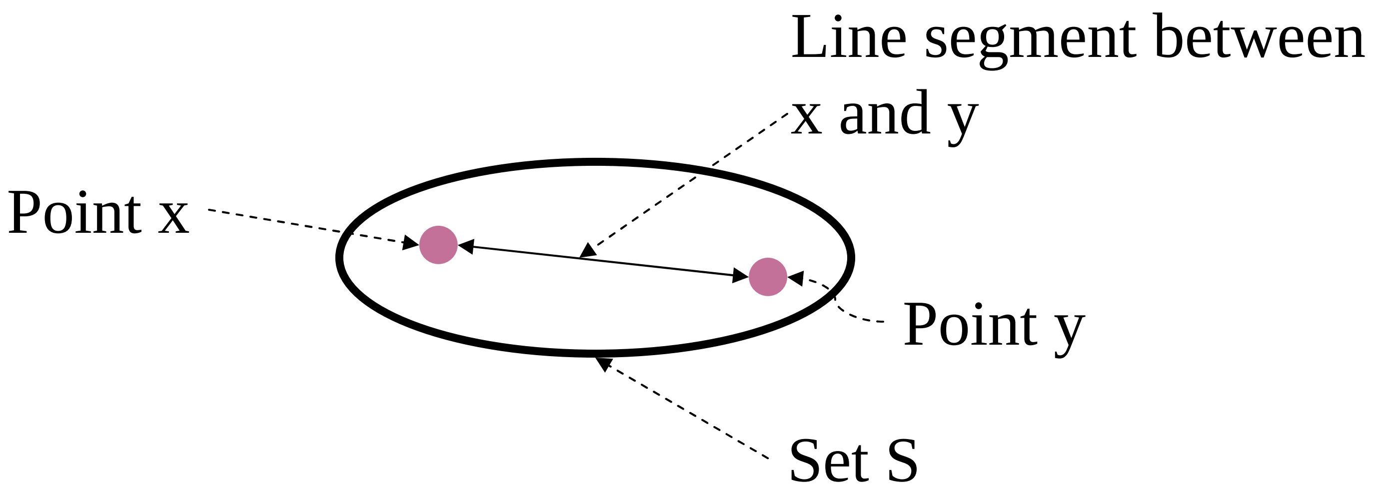
- This problem is the same as:
- Maximize f(x) over all x in the set {x | g(x) = < c}

- The set {x | g(x) = < c} is called the **feasible region.**

Definitions: Convex Set

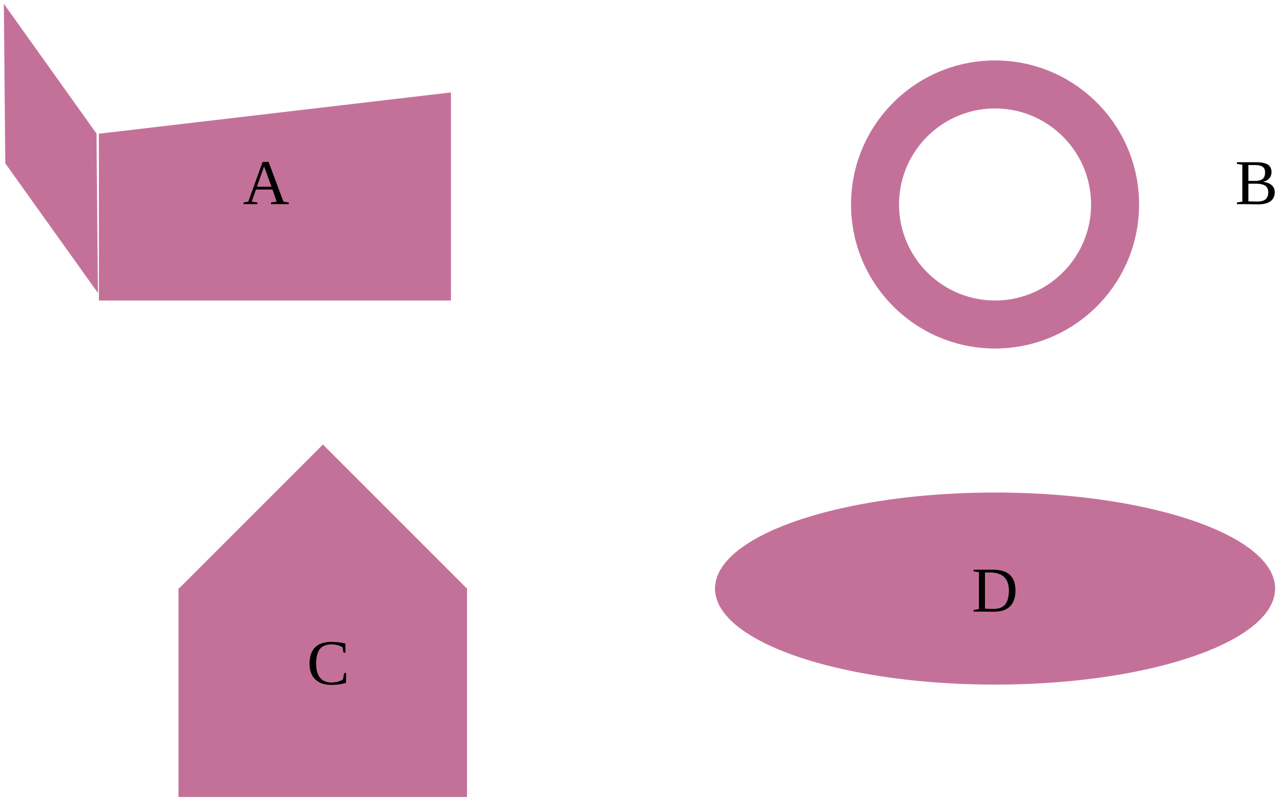
Convex set is a set of points in a vector space such that for any two points x and y in the set, the line segment between the two points is also in the set.

Definitions: Convex Set

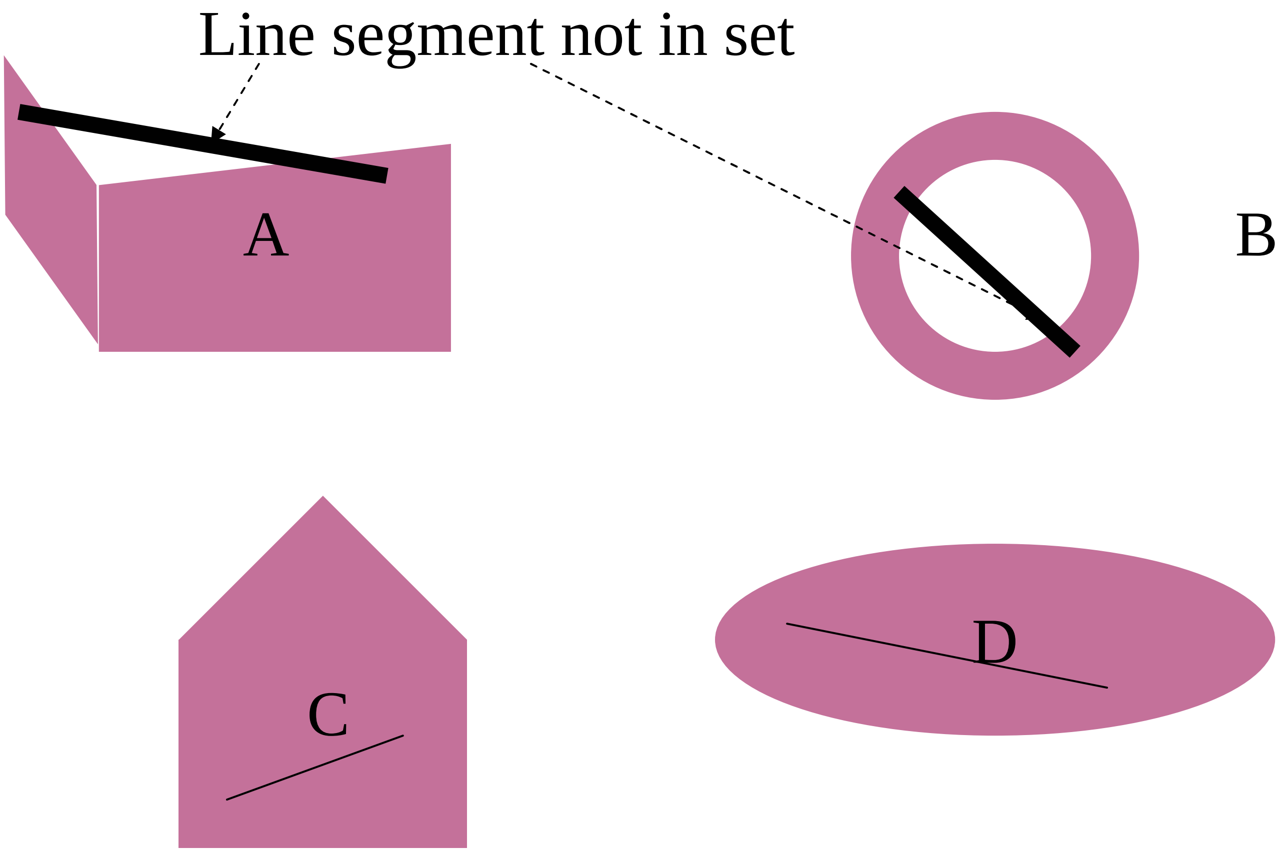


The line segment between any two points x and y in set S lies within set S; hence set S is convex.

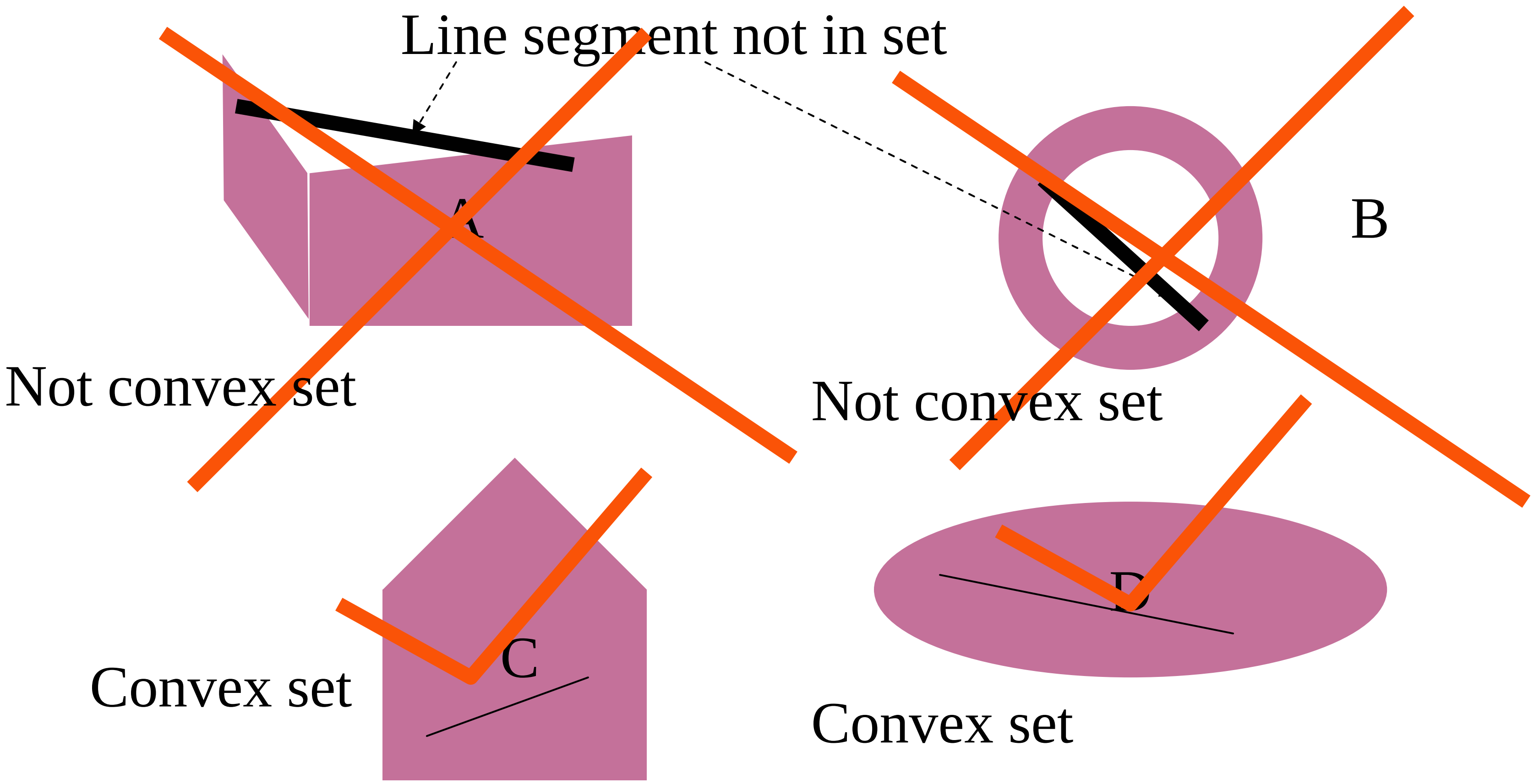
Problems: Which Sets are Convex?



Problems: All Line Segments Lie in Set?



Problems: Line Segments



Definitions: Convex Set

Convex set is a set of points in a vector space such that for any two points x and y in the set, the line segment between the two points is also in the set.

For all x, y in set S: For all r where 0 =< r =< 1:
(1-r).x + r.y is in S

As r varies from 0 to 1, the point (1-r).x + r.y traverses the line segment from x to y.

Each point in the line segment can be represented as (1-r).x + r.y for some value of r.

Definition: Convex Combination

- A point q is in a convex combination of a set of points $p_1, p_2, \ldots, p_k$ if and only if there exists non-negative numbers (scalars) $r_1, r_2, \ldots, r_k$ such that:

q = Sum over j of $p_j * r_j$

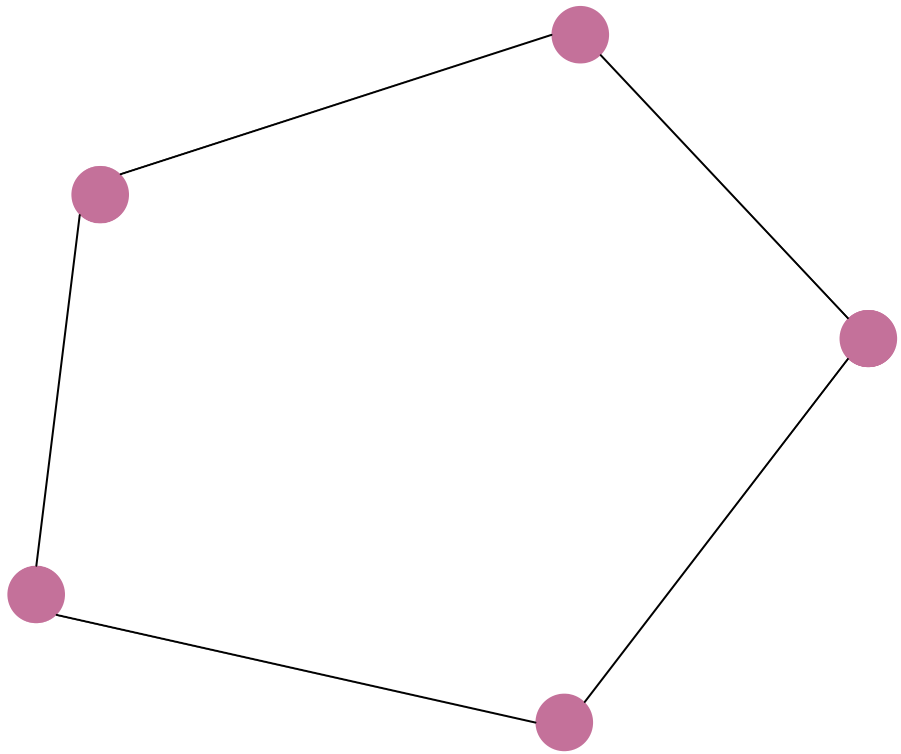
And

Sum over j of $r_j = 1$

Example: Convex Combination

What is the set of all points that are convex combinations of these 5 points?

Example: Convex Combination



What is the set of all points that are convex combinations of these 5 points?

It is the set of points within this polygon.

Theorems about Convex Sets

- A set S is a convex set if and only if every convex combination of any (non-empty) set of points in S is also in S.
- Note that the points on the line segment between points x and y are convex combinations of x and y.

Point of Closure

- A point p is a ***point of closure*** of a set S if and only if every ball of *<u>positive</u>* radius with center p has a non-empty intersection with set S.

- Examples: What are the points of closure for the following sets?
  - 0 =< x < 2
  - 0 =< x =< 2
  - x*x + y*y < 4
  - x*x + y*y =< 4

Point of Closure

- Examples: What are the points of closure for the following sets?
  - 0 =< x < 2 → 0 =< x =< 2
  - 0 =< x =< 2 → 0 =< x =< 2
  - x*x + y*y < 4 → x*x + y*y =< 4
  - x*x + y*y =< 4 → x*x + y*y =< 4

Definition: Closed Set

- A set is closed if and only if it includes all its points of closure.

- Examples: Which of the following sets are closed?
  - 0 =< x < 2
  - 0 =< x =< 2
  - x*x + y*y < 4
  - x*x + y*y =< 4

Definition: Closed Set

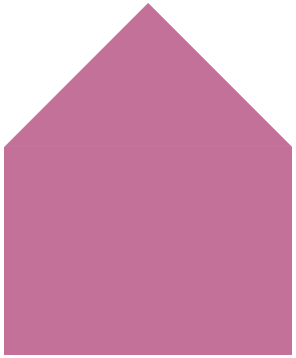
- A set is closed if and only if it includes all its points of closure.

- Examples: Which of the following sets are closed?
  - 0 =< x < 2 ➔ NOT CLOSED
  - 0 =< x =< 2 ➔ CLOSED
  - x*x + y*y < 4 ➔ NOT CLOSED
  - x*x + y*y =< 4 ➔ CLOSED

Definition: Extreme Point

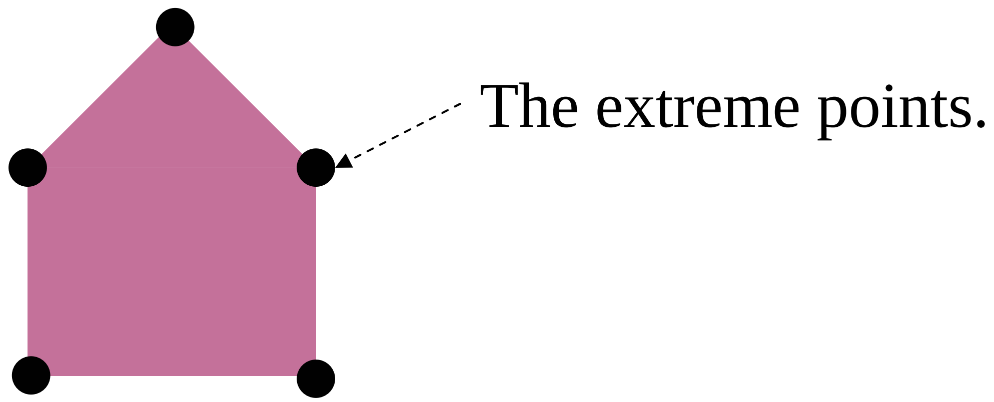
- A point v is an **extreme point** of a set S (of points in a vector space) if and only if:
  - v is in S
  - There do not exists points u and w, distinct from v, where:
    - u and w are in S, and
    - the line segment between u and w passes through v.

Extreme Points



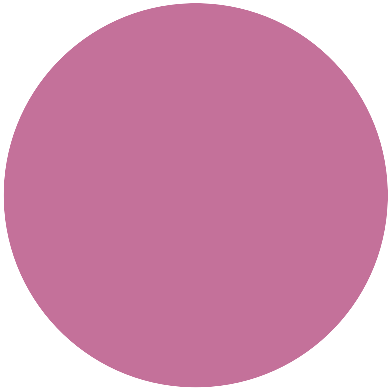
What are the extreme points?

Extreme Points



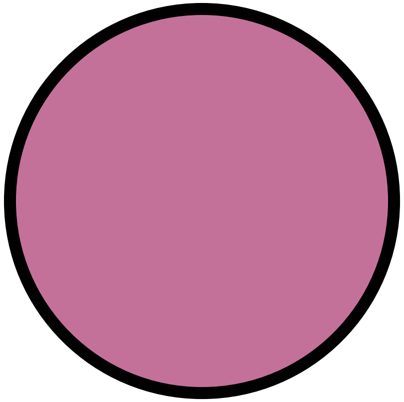
Example: Extreme Points

What are extreme points of the disk?



Example: Extreme Points

Extreme points of the disk are the set of points on the circumference



Extreme Points: Examples

- What are extreme points of the following sets?
- {x | 0 =< x =< 4}
- { x | x*x + y*y =< 16 }
- { (x,y) | 0 =< x =< 1 AND 0 =< y =< 1}

Extreme Points: Examples

- What are extreme points of the following sets?
- {x | 0 =< x =< 4}
  - **x = 0 and x = 4**
- { x | x*x + y*y =< 16 }
  - **{ (x,y) | x*x + y*y = 16 }**
- { (x,y) | 0 =< x =< 1 AND 0 =< y =< 1}
  - **{(x,y) = (0,0) or (0,1) or (1,0) or (1,1) }**

Definition: Bounded Set

- A set is bounded if and only if there exists a ball with finite radius such that the set is included within the ball.

- Examples: Are the following sets bounded?
- $\{x \mid x >= 0\}$
- $\{ (x,y) \mid x >= 0 \text{ and } y >= 0 \text{ and } x + y =< 1 \}$

Definition: Bounded Set

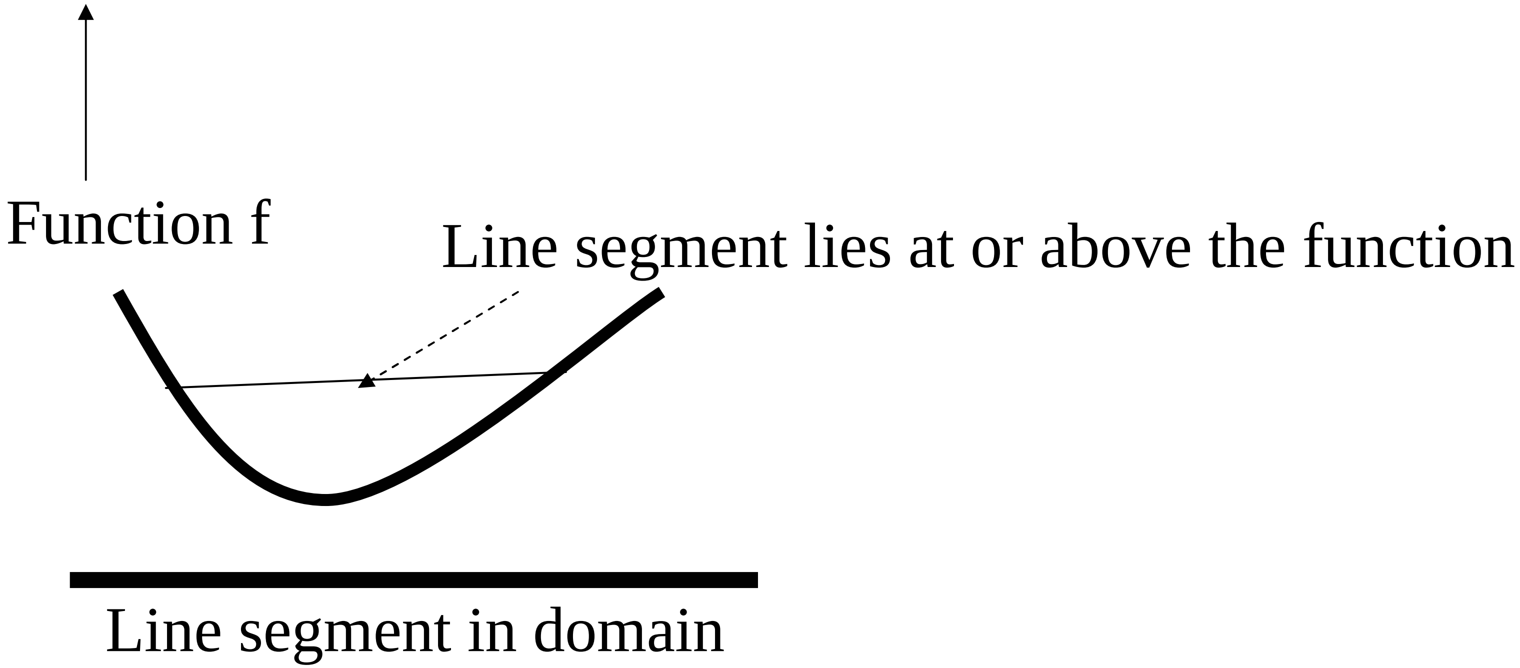
- A set is bounded if and only if there exists a ball with finite radius such that the set is included within the ball.

- Examples: Are the following sets bounded?
- {x | x >= 0}
  - **No, because x can become arbitrarily large**
- { (x,y) | x >= 0 and y >= 0 and x + y =< 1 }
  - **Yes, because this set is included within a ball with center (0,0) and radius 1.**

Theorem

- A closed bounded convex set S is the set of points that are convex combinations of the extreme points of S.

Definition: Convex Function

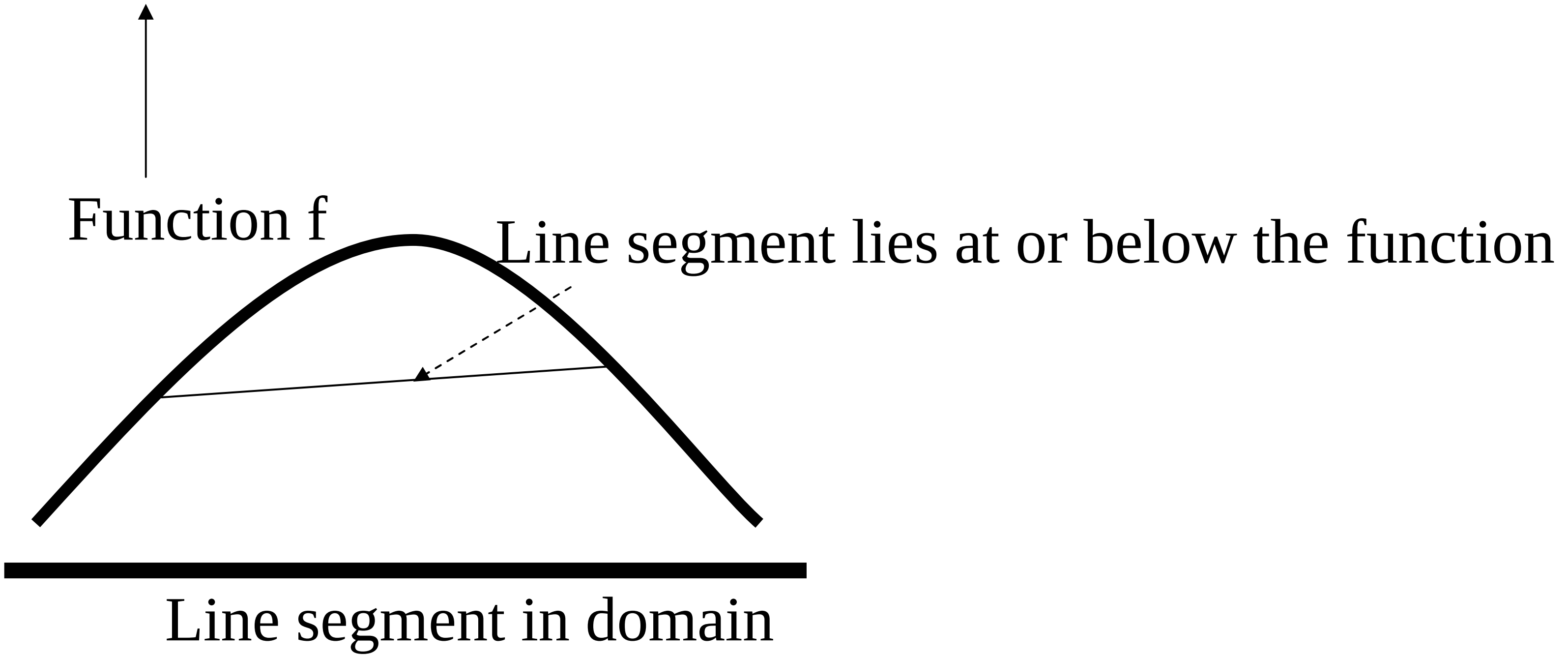


Definition: Convex Function

- Let f be a function defined over a convex set.
- f is convex if and only if, for every two points x_1, x_2 in the domain of f and for any non-negative scalars r_1 and r_2 that sum to 1:

$$r_1 . f(x_1) + r_2 . f(x_2) \geq f(r_1 * x_1 + r_2 * x_2)$$

Definition: Concave Function

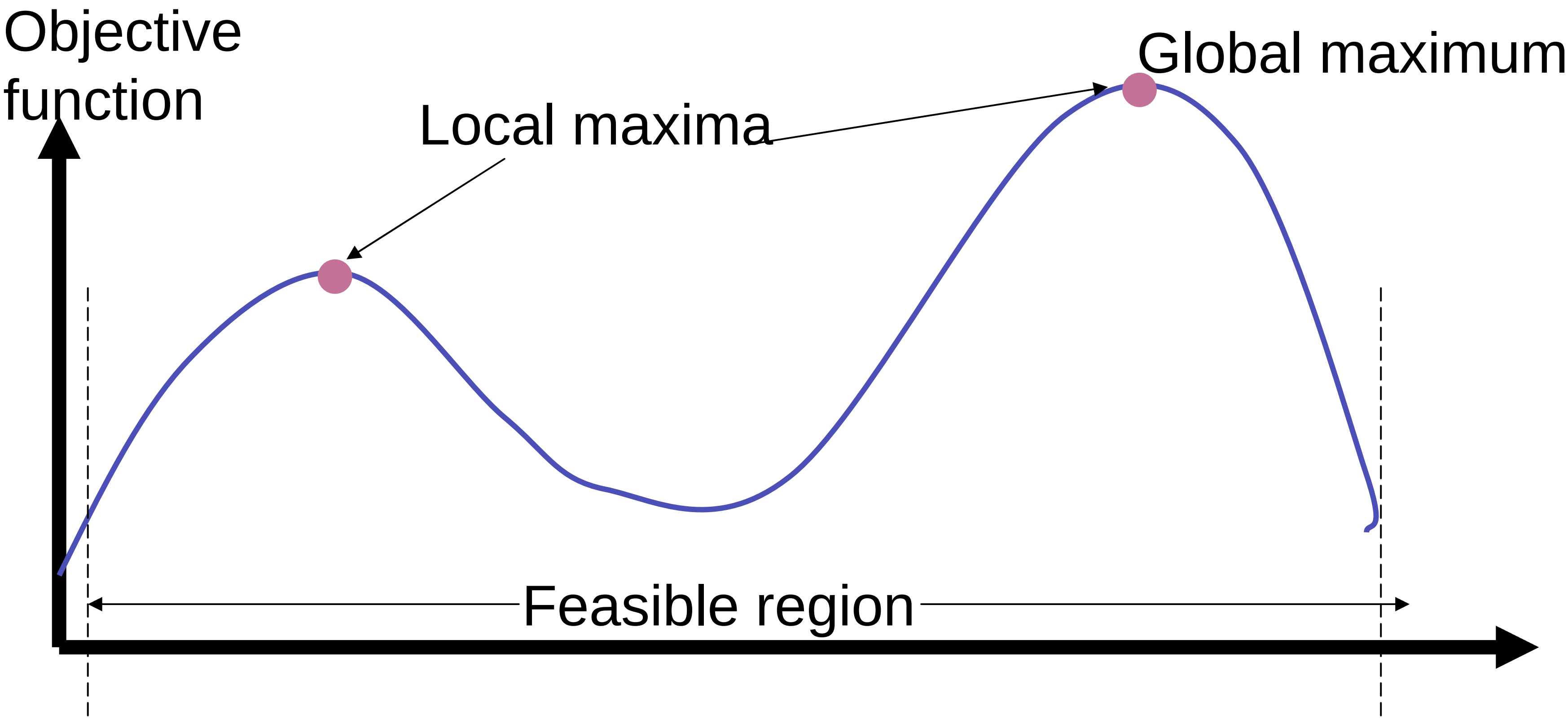


Definition: Concave Function

- Let f be a function defined over a convex set.
- f is concave if and only if, for every two points x_1, x_2 in the domain of f and for any non-negative scalars r_1 and r_2 that sum to 1:

$$r_1 . f(x_1) + r_2 . f(x_2) =< f(r_1 * x_1 + r_2 * x_2)$$

Local and Global Maxima



Local Maximum

- Given a problem, maximize f(x) subject to x in set S,
- Where set S is the feasible region.

- A point p in S is a local maximum for this problem if and only if there exists a ball with positive radius centered at p such that for all points q in the ball and in S:

$$f(p) >= f(q)$$

Global Maximum

- Given a problem, maximize f(x) subject to x in set S,
- Where set S is the feasible region.

- A point p in S is a global maximum for this problem if and only if for all points q in S:

$$f(p) >= f(q)$$