CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. (a) Let n_a the number of a signals and n_b the number of b signals. Let $p = \Pr\{a|A\}$. With slight abuse of notation we denote by A the software design and the event that A occurs. This is done only for simplifying the notation.

We have that

$$\Pr\{A|n_a, n_b\} = \frac{\Pr\{n_a, n_b|A\} \cdot \Pr\{A\}}{\Pr\{n_a, n_b|A\} \cdot \Pr\{A\} + \Pr\{n_a, n_b|B\} \cdot \Pr\{B\}}$$

Expanding the expression we obtain

$$\frac{\binom{n_a+n_b}{n_a} \cdot p \cdot (1-p)^{n_b} \cdot \Pr\{A\}}{\binom{n_a+n_b}{n_a} p^{n_a} \cdot (1-p)^{n_b} \cdot \Pr\{A\} + \binom{n_a+n_b}{n_b} \cdot (1-p)^{n_a} \cdot p^{n_b} \cdot P\{B\}}$$

Let diff $= n_a - n_b$. After simplification, using the fact that $\Pr\{A\} = \Pr\{B\} = 0.5$ the expression above becomes

$$\Pr\{A|n_a, n_b\} = \frac{p^{\text{diff}}}{p^{\text{diff}} + (1-p)^{\text{diff}}}$$

Using symmetry of the binomial distribution, we obtain

$$\Pr\{B|n_a, n_b\} = \frac{(1-p)^{\text{diff}}}{p^{\text{diff}} + (1-p)^{\text{diff}}}$$

It is easy to see that $\Pr\{A|n_a, n_b\} \geq \Pr\{B|n_a, n_b\}$ iff $(\frac{p}{1-p})^{\text{diff}} > 1$ with equality when $\text{diff} = 0$. This leads to the following decisions:

- $n_a > n_b \Rightarrow A$
- $n_a = n_b \Rightarrow A$ with probability $\frac{1}{2}$ and B with probability $\frac{1}{2}$
- $n_a < n_b \Rightarrow B$

Next we wish to evaluate the optimum N. First we show that all even values of N can be discarded because the probability of a correct decision with N even ($N > 0$) is the same as the probability of a correct decision with $N-1$ signals. For example, the probability of making a correct decision is the same for 1 and 2 signals; for 3 and 4; for 5 and 6 and so on. Generally speaking, let N be an odd number denoting the number of signals. Then the following identities hold

$$\Pr\{\text{decision} = A|N \text{ signals }\} = \Pr\{n_a > \frac{N}{2}|N \text{ signals }\} =$$
$$\Pr\{n_a > \frac{N+1}{2}|N+1 \text{ signals }\} + \frac{1}{2} \cdot \Pr\{n_a = \frac{N+1}{2}|N+1 \text{ signals }\} =$$
$$\Pr\{\text{decision} = A|(N+1) \text{ signals }\}$$

We have the decision tree shown in figure 1

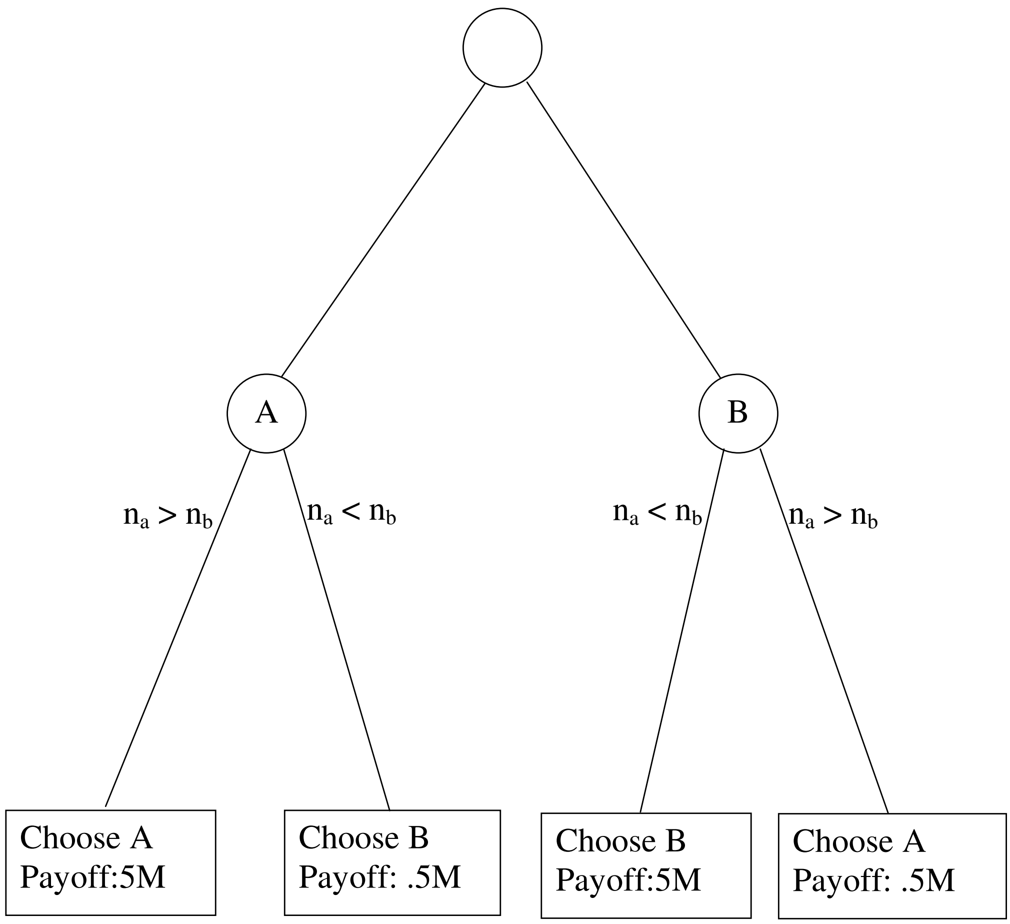

Figure 1: Decision tree

Denote by Q the probability that $n_a > \frac{N}{2}$ conditioned to the fact that the state is A. It is easy to see that

$$Q = \sum_{n_a=\frac{N+1}{2}}^{N} \binom{N}{n_a} \cdot p^{n_a} \cdot (1-p)^{N-n_a}$$

The expected profit is then given by $E[\text{profit}|N \text{ signals}] = Q \cdot \$5,000,000 + (1-Q) \cdot \$500,000 - N \cdot 10,000$

Since it is hard to compute analytically the number of signals which maximizes the expected profit, the following Matlab program has been used

```
function Evaluate()

p =  .6; %p is Pr{a|A}
m = 100; %max number of evaluations
profit = zeros(1,m); %profit(i) will contain the expected profit for i signals

i = 1;

while (i<=m)

 %Compute binomial distribution
  s = 0;
  for k = (i+1)/2:i
   s = s + (nchoosek(i,k)) * p^k * (1-p)^(i-k);
  end

  profit(i) = s * 5000000 + (1-s) * 500000;
  %profit without considering the cost of signals

  profit(i) = profit(i) - (i * 10000);
  %profit after detracting the cost of the signals

  i = i + 2;

end

plot([1:2:m],profit(1:2:m)); %Draw the profit function
```

The resulting plot is illustrated in figure 2. It is clear that the concave function achieves its maximum when $N = 47$.

It is interesting to notice how the optimal number of signals to buy varies as the conditional probability p changes. This is illustrated by means of the figure 3.

Figure 4 indicates how the optimal number of signals varies when the amount of prize if you guess right increases. Such prize ranges from 500,000 (equal to the amount you get

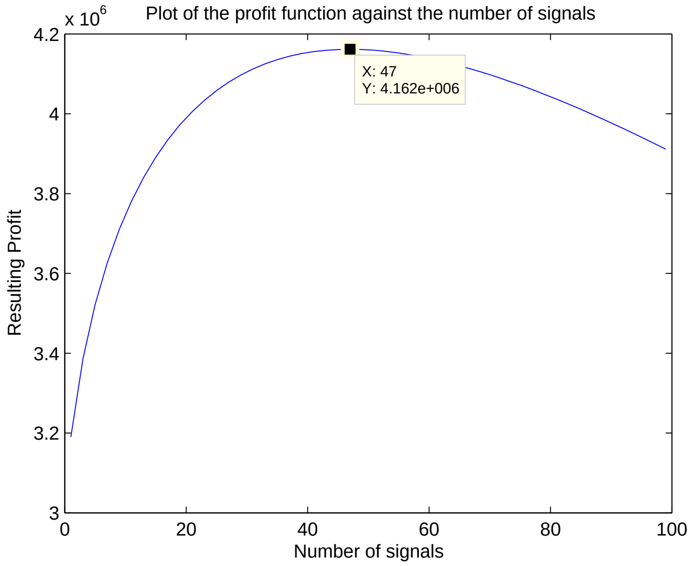

Figure 2: Plot of the expected profit function for a number of N signals, p = 0.6

if you guess wrong) to 10,000,000. Three different curves are plotted, each corresponding with a different value of the conditional probability p.

(b) If I do not buy any signal, I would choose indifferently for A or B and the expected profit would be $\frac{\$5{,}500{,}000}{2} = 2,275,000$. If I buy one signal, then the expected profit is given by

$$E[\text{Profit}] \quad = \quad 0.6 \cdot \$5,000,000 + 0.4 \cdot \$500,000 - \$10,000 = \$3,190,000$$

Since the expected profit after buying one signal is higher than the expected profit without buying any signal, I would not stop and buy the signal. Doing calculation it can be seen that it is always better to buy another signal regardless the first received signal is a or b. If the second signal is the same as the first then I would stop. I would then choose A if both signals are a's and B if both signals are b's. If the second received signal is different from the first received signal, then I would still buy another signal. It is difficult to figure out a general decision rule for the problem since the number of possibilities grow exponentially as the number of signal increases. However, the method to compute the optimal number of signals incrementally consists in drawing the bayesian decision tree and extend it every time a new signal is received.

2. (a) The first price auction and the second price auction have the same expected revenue. Denoting by V_1, V_2 the two r.v. uniformly distributed over the interval $[0, 100]$, we have that the pdf of $V_{\max} = \max\{V_1, V_2\}$ is

$$p(x) \quad = \quad \frac{d\,\Pr\{V_{\max} \leq x\}}{dx} = \frac{x}{5000}$$

where

$$\Pr\{V_{\max} \leq x\} \quad = \quad \Pr\{V_1 \leq x\} \cdot \Pr\{V_2 \leq x\} = \left(\frac{x}{100}\right)^2$$

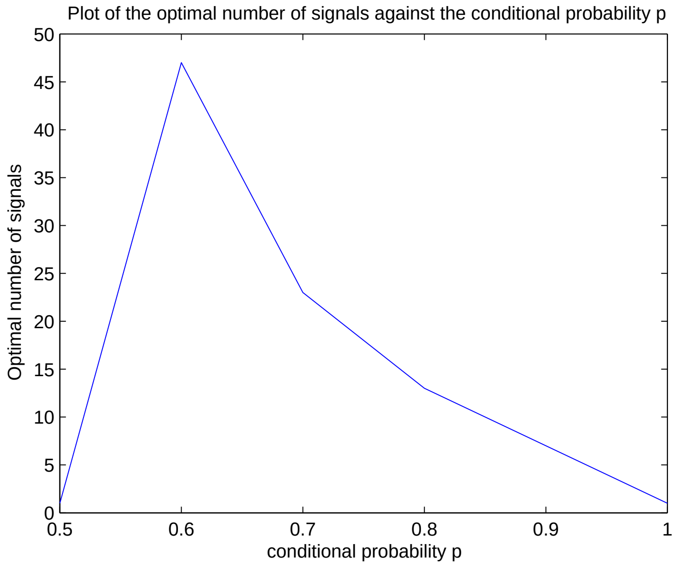

Figure 3: Plot of the optimal number of signals as a function of the conditional probability p

while the pdf of $V_{\min} = \min\{V_1, V_2\}$ is

$$p(x) = \frac{d\Pr\{V_{\min} < x\}}{dx} = \frac{1}{50} \cdot \left(1 - \frac{x}{100}\right)$$

where

$$\begin{aligned}\Pr\{V_{\min} < x\} &= 1 - \Pr\{V_{\min} \geq x\} = 1 - \Pr\{V_1 \geq x, V_2 \geq x\} = \\ &= 1 - \Pr\{V_1 \geq x\} \cdot \Pr\{V_2 \geq x\} = 1 - (1 - \Pr\{V_1 < x\})^2 = \\ &= 1 - \left(1 - \frac{x}{100}\right)^2\end{aligned}$$

The expected value of $V_{\max}$ is

$$E[V_{\max}] = \frac{1}{5000}\int_0^{100} x^2 dx = \frac{200}{3}$$

while the expected value of $V_{\min}$ is

$$E[V_{\min}] = \frac{1}{5000}\int_0^{100} (100x - x^2)dx = \frac{100}{3}$$

The dominated strategy (and Bayesian Nash equilibrium) for the bidders in a first price action is to bid $\frac{N-1}{N} \cdot V_i$, i.e. $\frac{V_i}{2}$ since there are only two players (see class notes for a proof). Therefore, the expected value of the first price auction is half of the expected value of $V_{\max}$, i.e $\frac{100}{3}$. The dominated strategy in a second price action instead is for each player to bid the real value. Therefore, the expected value of the second price auction is $E[V_{\min}] = \frac{100}{3}$

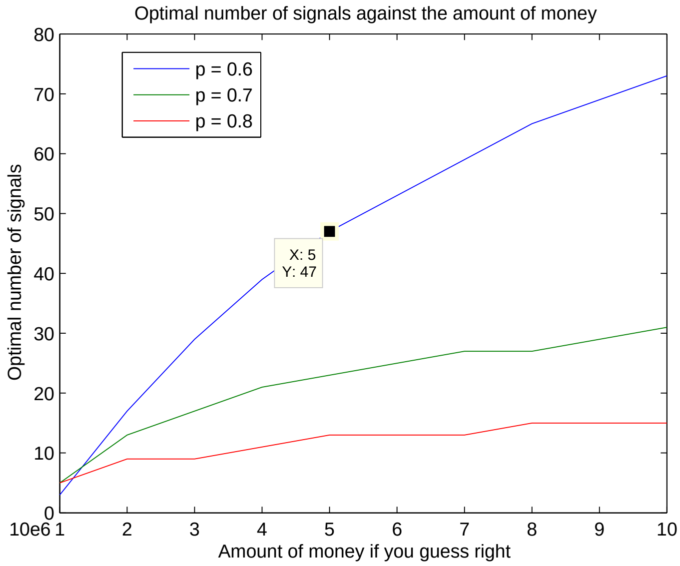

Figure 4: Plot of the optimal number of signals against the prize you get if you guess right. Such prize ranges from 500,000 to 10,000,000. The amount you get if you guess wrong is fixed to 500,000.

(b) Denoting by R the reserve price, we have that the expected revenue is:

$$\begin{aligned} E[R] &\geq \Pr\{V_{\max} \geq R\} \cdot R = (1 - \Pr\{V_{\max} \leq R\}) \cdot R = \\ &= \left[1 - \left(\frac{R}{100}\right)^2\right] \cdot R \end{aligned}$$

The value that maximizes the expression on the right hand side of the equality is $\frac{100\sqrt{3}}{3}$. Setting $R = \frac{100\sqrt{3}}{3}$ in the expression on the right hand side we obtain $\frac{200\sqrt{3}}{9}$. This value is bigger than $\frac{100}{3}$. It follows that $E[R] > \frac{100}{3}$.