CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. Assume the $\$300,000$ dollars is in Room 1 (R_1).

Game 1: Assuming that Monty Hall opens R_2 if Player chooses R_1, we have that

	Case 1	Case 2	Case 3
Player	R_1	R_2	R_3
Monty Hall	R_2	R_3	R_2
Player	R_3	R_1	R_1
Outcome	\$0	\$300,000	\$300,000

So, if player chooses R_2 or R_3 then he wins, otherwise he loses:

$$E[\text{Game 1}] = 2 \cdot \frac{1}{3} \cdot 300,000 + \frac{1}{3} \cdot 0 = 200,000$$

Game 2: Assuming again that Monty Hall opens R_2 if Player chooses R_1.

	Case 1	Case 2	Case 3
Player	R_1	R_2	R_3
Monty Hall	R_2	R_3	R_2
Player	R_1	R_2	R_3
Outcome	\$300,000	\$0	\$0

So, only if player chooses R_1, he wins:

$$E[\text{Game 2}] = \frac{1}{3} \cdot 300,000 + 2 \cdot \frac{1}{3} \cdot 0 = 100,000$$

Game 3: Assuming again that Monty Hall opens R_2 if Player chooses R_1, we have that

	Case 1a	Case 1b	Case 2a	Case 2b	Case 3a	Case 3b
Player	R_1		R_2		R_3	
Monty Hall	R_2		R_3		R_2	
Player	R_1	R_3	R_2	R_1	R_3	R_1
Outcome	\$300,000	\$0	\$0	\$300,000	\$0	\$300,000

Denoting by P_1 the random variable representing the first choice of Player and by P_2 the random variable representing the second choice of Player, we obtain

$$
\begin{aligned}
E[\text{Game 3}] &= \Pr\{\text{win in Game 3}\} \cdot 300,000 = \\
&= (\Pr\{P_1 = R_1, P_2 = R_1\} + 2 \cdot \Pr\{P_1 = R_2, P_2 = R_1\}) \cdot 300,000 = \\
&= \left(\frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} \cdot \frac{1}{2}\right) \cdot 300,000 = \frac{1}{2} \cdot 300,000 = \\
&= 150,000
\end{aligned}
$$

2. (a) \$25 dollars is the cost of the game and Player 1 wins \$100 if both cards are kings.

   – **No bribe**

$$E[\text{value}] = 100 \cdot \Pr\{C_1 = K, C_2 = K\} - 25 = -3.57$$

where

$$\Pr\{C_1 = K, C_2 = K\} = \Pr\{C_1 = K\} \cdot \Pr\{C_2 = K | C_1 = K\} = \frac{3}{14}$$

and

$$\Pr\{C_1 = K | C_2 = K\} = \Pr\{C_2 = K | C_1 = K\} = \frac{\Pr\{C_1 = K, C_2 = K\}}{\Pr\{C_2 = K\}} = \frac{3}{7}$$

Bribe 1

$$E[\text{value}] = \frac{1}{2} \cdot (100 \cdot \Pr\{C_2 = K | C_1 = K\} - 25) - 6 = 2.93$$

Bribe 2

$$E[\text{value}] = \frac{1}{2} \cdot (100 \cdot \Pr\{C_1 = K | C_2 = K\} - 25) - 5 = 3.93$$

Bribe 3

$$
\begin{aligned}
E[\text{value}] &= \Pr\{C_1 = K, C_2 = K | C_1 = K \vee C_2 = K\} \cdot \\
&\quad 100 - 25 - 4
\end{aligned}
$$

where

$$
\begin{aligned}
\Pr\{C_1 = K \vee C_2 = K\} &= \Pr\{C_1 = K\} + \Pr\{C_2 = K\} - \Pr\{C_1 = K, C_2 = K\} = \\
&= 1 - \frac{3}{14} = \frac{11}{14}
\end{aligned}
$$

and

$$
\begin{aligned}
\Pr\{C_1 = K, C_2 = K | C_1 = K \vee C_2 = K\} &= \frac{\Pr\{C_1 = K, C_2 = K\}}{\Pr\{C_1 = K \vee C_2 = K\}} = \\
&= \frac{3}{14} \cdot \frac{14}{11} = \frac{3}{11}
\end{aligned}
$$

Therefore $E[\text{value}] = \frac{11}{14} \cdot (\frac{3}{11} \cdot 100 - 25) - 4 = -2.21$

- **Bribe 4** We have

$$\begin{aligned}\Pr\{C_1 = K, C_2 = K | C_1 \text{ reported } K\} &= \frac{\Pr\{C_1 = K, C_2 = K, C1 \text{ reported } K\}}{\Pr\{C_1 \text{ reported } K\}} = \\ &= \frac{\frac{3}{14} \cdot \frac{9}{10}}{\frac{1}{2} \cdot \frac{9}{10} + \frac{1}{2} \cdot \frac{1}{10}} = 0.38\end{aligned}$$

and

$$\begin{aligned}\Pr\{C_1 = K, C_2 = K | C_1 \text{ reported } Q\} &= \frac{\Pr\{C_1 = K, C_2 = K, C1 \text{ reported } Q\}}{\Pr\{C_1 \text{ reported } Q\}} = \\ &= \frac{\frac{3}{14} \cdot \frac{1}{10}}{\frac{1}{2} \cdot \frac{9}{10} + \frac{1}{2} \cdot \frac{1}{10}} = 0.04\end{aligned}$$

After observing that the expected profit when the first card is reported to be a queen is negative, namely 100 * 0.04 -25, we decide to play only when the first card is reported to be a king. Under this choice the expected profit is $E[\text{value}] = \Pr\{C_1 \text{ reported } K\} \cdot (100 \cdot 0.38 - 25) - 3 = 3.79$

- **Bribe 5** Let D be the event that one card is a king of diamond. We have

$$\Pr\{C_1 = K, C_2 = K | D\} = \frac{\Pr\{C_1 = K, C_2 = K, D\}}{\Pr\{D\}} = \frac{\frac{1}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{1}{7}}{\frac{1}{8} + \frac{7}{8} \cdot \frac{1}{7}} = 0.42$$

and

$$\Pr\{C_1 = K, C_2 = K | \bar{D}\} = \frac{\Pr\{C_1 = K, C_2 = K, \bar{D}\}}{\Pr\{\bar{D}\}} = \frac{\frac{3}{8} \cdot \frac{2}{7}}{\frac{7}{8} \cdot \frac{6}{7}} = 0.14$$

After observing that the expected profit when no card is a king of diamond is negative, namely $100 \cdot 0.14 - 25$, we decide to play when at least one card is a king of diamonds. Under this choice the expected profit is $E[\text{value}] = \Pr\{D\} \cdot (100 \cdot 0.42 - 25) - 2 = 2.46$

- **Bribe 6** Let R be the event that one card is a red king (notice there are at most two red kings). We have

$$\Pr\{C_1 = K, C_2 = K | R\} = \frac{\Pr\{C_1 = K, C_2 = K, R\}}{\Pr\{R\}} = \frac{\frac{1}{2} \cdot \frac{2}{7} + \frac{1}{4} \cdot \frac{1}{7}}{\frac{1}{4}\frac{1}{7} + \frac{1}{4} \cdot \frac{6}{7} + \frac{3}{4}\frac{2}{7}} = 0.38$$

and

$$\Pr\{C_1 = K, C_2 = K|\bar{R}\} = \frac{\Pr\{C_1 = K, C_2 = K, \bar{R}\}}{\Pr\{\bar{R}\}} = \frac{\frac{1}{4}\cdot\frac{1}{7}}{\frac{6}{8}\cdot\frac{5}{7}} = 0.05$$

After observing that the expected profit when no card is a red king is negative, namely $100 \cdot 0.05 - 25$, we decide to play only when we have the information that one card is a red king. Under this choice, the expected profit is $E[\text{value}] = \Pr\{D\} \cdot (100 \cdot 0.38 - 25) - 1 = 5.25$.

Hence, the optimal strategy is to buy Bribe 6 and play only if one of the two card is a red king. (see table below)

No Bribe	-3.57
Bribe 1	2.93
Bribe 2	3.93
Bribe 3	-2.21
Bribe 4	3.79
Bribe 5	2.46
Bribe 6	5.25

Table 1: the expected value of each strategy

(b) Table above shows that the information "one card is a king of diamond" is less profitable that the information "one of the two cards is a red king". However, knowing that one card is a king of diamond seem to convey more information than simply knowing that one of the two cards is a red king. In other circumstances, e.g. cases 1,2,3, having more information like knowing exactly whether the first or the second card is a king yields more revenue than simply knowing that one of the two cards is a king.

(a) The probability of oil on the rance is 0.01, i.e.

$$\Pr\{\text{oil}\} = 0.01$$

A seismic test can be done, its cost is \$100,000 and it gives correct results with probability 0.9 (and incorrect results with probability 0.1):

$$\Pr\{\text{positive}|\text{oil}\} = 0.9$$

while

$$\Pr\{\text{positive}|\text{no oil}\} = 0.1$$

and, therefore,

$$\Pr\{\text{negative}|\text{oil}\} = 0.1$$

and

$$\Pr\{\text{negative}|\text{no oil}\} = 0.9$$

The cost of a drill is \$1,000,000 and the probability that the drill will produce oil assuming that there is oil on the ranch is 0.95,

$$\Pr\{\text{gusher}|\text{oil}\} = 0.95$$

and

$$\Pr\{\text{dry}|\text{oil}\} = 0.05$$

while it is 0 if there is no oil on the ranch:

$$\Pr\{\text{gusher}|\text{no oil}\} = 0$$

The value of a gusher is \$20,000,000 and the one of a dry hole is \$0.
The probability that the test gives correct results is

$$\begin{aligned} \Pr\{\text{positive}\} &= \Pr\{\text{positive}|\text{oil}\} \cdot \Pr\{\text{oil}\} + \Pr\{\text{positive}|\text{no oil}\} \cdot \Pr\{\text{no oil}\} = \\ &= 0.9 \cdot 0.01 + 0.1 \cdot 0.99 = 0.108 \end{aligned}$$

while the probability that the test gives negative result is

$$\begin{aligned} \Pr\{\text{negative}\} &= \Pr\{\text{negative}|\text{oil}\} \cdot \Pr\{\text{oil}\} + \Pr\{\text{negative}|\text{no oil}\} \cdot \Pr\{\text{no oil}\} = \\ &= 0.1 \cdot 0.01 + 0.9 \cdot 0.99 = 0.892 \end{aligned}$$

The probability that the drill will produce oil is

$$\Pr\{\text{gusher}\} = \Pr\{\text{oil}\} \cdot \Pr\{\text{gusher}|\text{oil}\} = 0.0095$$

The probability that the drill will produce oil assuming the test is positive

$$\begin{aligned} \Pr\{\text{gusher}|\text{positive}\} &= \Pr\{\text{gusher}|\text{oil}\} \cdot \Pr\{\text{oil}|\text{positive}\} = \\ &= 0.95 \cdot 0.083 = 0.079 \end{aligned}$$

where the $\Pr\{\text{oil}|\text{positive}\}$ can be obtained applying the Bayes Rule; while the probability that the drill will produce oil assuming the test is negative is

$$\begin{aligned} \Pr\{\text{gusher}|\text{negative}\} &= \Pr\{\text{gusher}|\text{oil}\} \cdot \Pr\{\text{oil}|\text{negative}\} = \\ &= 0.95 \cdot 0.001 = 0.00095 \end{aligned}$$

Now, consider the cases in Table **??** obtained from the decision tree in Figure **??** where for the case **ACG** (no test and drill) we get

$$20,000,000 \cdot \Pr\{\text{gusher}\} - 1,000,000 = -810,000$$

	ACF	**ACG**	**ABDG**	**ABDH**	**ABEL**	**ABEM**
Test	no	no	yes	yes	yes	yes
Result			negative	negative	positive	positive
Drill	no	yes	no	yes	no	yes
Outcome	0	-800,000	-100,000	- 1,081,000	-100,000	480,000

Table 2: Table for the game in Exercise 3(a)

the case **ABDH** (negative test and drill) we get

$$20,000,000 \cdot \Pr\{\text{gusher}|\text{negative}\} - 100,000 - 1,000,000 = -1,081,000$$

and for the case **ABEM** (positive test and drill)

$$20,000,000 \cdot \Pr\{\text{gusher}|\text{positive}\} - 100,000 - 1,000,000 = 480,000$$

After I have tested and gotten a positive, drilling is my choice with an expected value of $580,000 = 0.079 * 20,000,000$ - $1,000,000$. After I have tested and get a negative I do not drill, for an expected value of 0. So since the probability of a positive is .108, the expected value of the test is .108*580,000, which is less than the price of the test. Therefore it is better not to test. The overall conclusion is that the best strategy is: neither testing nor drilling.

(b) I would not borrow money from the bank to conduct drilling or a seismic test. Though the amount of money that I would get from a gusher is high, the a-priori belief that there is oil on the ranch is too low. I would change my decision in one of the following cases:

- The a-priori belief is higher. For instance it is interesting to see what happens if the a-priori belief that there is oil is 0.1 or higher.
- The amount of money I would get in the case there is really oil is higher. It would be interesting to repeat the same calculations for the case when the prize is $200,000,000$. Increasing the probability that oil is found after drilling would not change my decision (even in the case that this probability is one).
  Acting as described above can be considered rational.

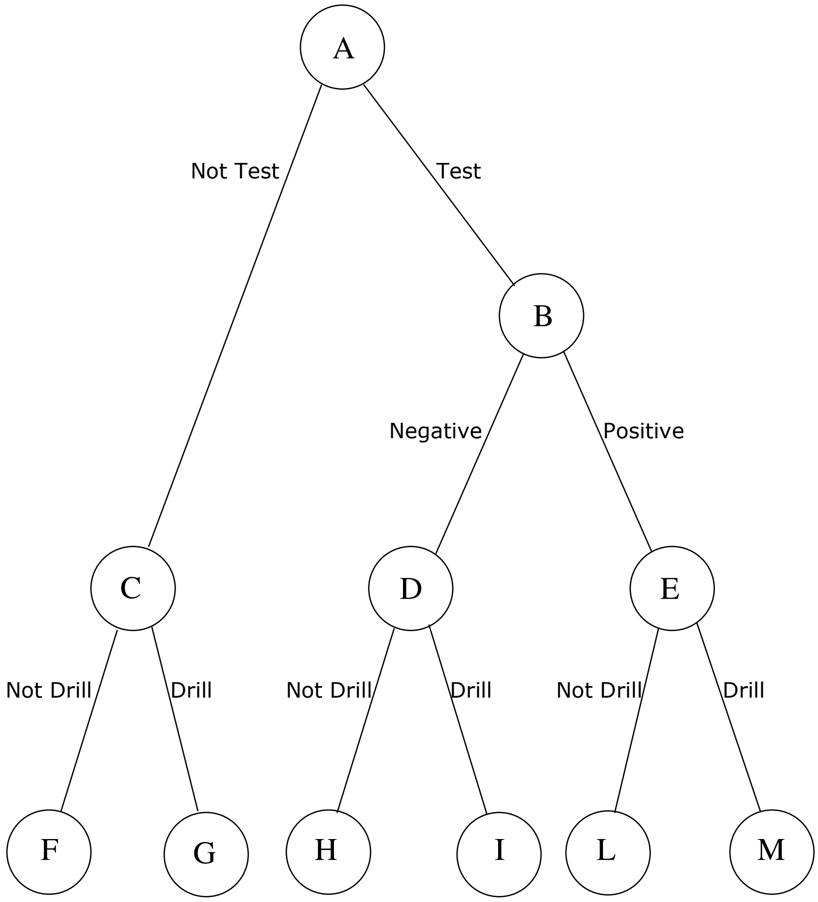

Figure 1: Tree for the game in Exercise 3(a)