CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. Using Bayes rule we have

$$\Pr(\text{coin } = A|\ h \text{ heads}) = \frac{\Pr(h \text{ heads} \mid \text{coin} = A) \cdot \Pr(\text{coin } = A)}{\Pr(h \text{ heads})} \tag{1}$$

Using the formula for the binomial coefficient to compute the probabilites above and using the data of the problem, we obtain

$$\frac{u\ \binom{h+t}{h}\ p^h\ (1-p)^t}{u\ \binom{h+t}{t}\ p^h\ (1-p)^t + (1-u)\ \binom{h+t}{t}\ q^h\ (1-q)^t} \tag{2}$$

After simplification, we obtain

$$\frac{u\ p^h\ (1-p)^t}{u\ p^h\ (1-p)^t + (1-u)\ q^h\ (1-q)^t} \tag{3}$$

If $q = 1 - p$, then $\Pr(\text{coin } = A|\ h \text{ heads})$ in equation 3 becomes

$$\frac{u\ p^h\ (1-p)^t}{u\ p^h\ (1-p)^t + (1-u)\ (1-p)^h\ p^t} \tag{4}$$

We can then rewrite the expression as

$$\frac{u\ p^h\ (1-p)^t}{p^h\ (1-p)^t\ [u + (1-u)\ p^{t-h}\ (1-p)^{h-t}]} \tag{5}$$

and after further simplification we obtain

$$\frac{u}{[u + (1-u)\ p^{t-h}\ (1-p)^{h-t}]} \tag{6}$$

A similar calculation can be done for computing $\Pr(\text{coin } = B|\ h \text{ heads})$. After doing it, we obtain

$$\frac{u}{[u + (1-u)\ (1-p)^{t-h}\ p^{h-t}]} \tag{7}$$

(i.) No, I would not buy the information (*ii.*) In order to compute the conditional probability of A given the experimental outcome "h heads and t tails", we only need to know the difference $h - t$, as it clearly appears from the formula in equations 6 and 7. Therefore, knowing the exact number of heads and tails in the experiment does not add any additional value.

(*iii.*) For any value of u, we can easily verify that $\Pr(A|x) > \Pr(B|x)$ iff

$$\left(\frac{1-p}{p}\right)^{t-h} > \left(\frac{1-p}{p}\right)^{h-t} \tag{8}$$

The expression 8 can be rewritten as

$$(1-p)^{2h-2t} > p^{2h-2t} \tag{9}$$

Since we are assuming $p > 0.5$ I would call head if $2h - 2t > 0$, i.e. the number of heads is larger than the number of tails.

(*iv.*) Yes, the answer to the previous questions would change. In the case when $p \neq q$ the probability of the next toss being h and t will depend on on the exact values of h and t and not only on their difference. Since the cost of the information is only one, one can easily check the expected profit after buying the information is higher than the expected profit without using any information.

(*v.*) In the case when $q = 1 - p$ and $u \neq 0.5$, the answer to first three previous questions does not change. The reason is that the criteria to decide if $\Pr(\text{head}|A) > Pr(\text{head}|B)$ is independent of u as it clearly appears from equation 9.

2. (a) The curve for the cost and utility functions is shown in figure 1:
Let $u_1(c) = 16\log(C)$ and $u_2(c) = 36\log(c)$. Furthermore, let $f(c) = x_1 + x_2 = 2c + 14c^2$ Picture 1 shows that the maximum M_C of the concave function $u_2(c) - f(c)$ is larger than the maximum m_c of the concave function $u_1(c) - f(c)$. Therefore, knowing that Computer Science is going to buy M_c units of capacity there will be no incentive to buy capacity for Economics. The consequence is that $x_1 = 0$. In order to find the price x_2 payed by Economics we have to determine c such that $\frac{\partial u_2}{c} = \frac{\partial f}{c}$. This follows from the fact that the function $36\log(c) - (2c + 14c^2)$ is concave. This equals to solving the second order equation $14c^2 + c - 18 = 0$ which implies $c = 1.0987$. Substituting this value into the expression $x_2 = 2c + 14c^2$, we obtain that $x_2 = 19.0987$.

(b) If Caltech is paying the entire amount, then Caltech will assign the capacity c which maximizes the concave function

$$(16 + 36)\log(c) - (2c + 14c^2) \tag{10}$$

Using the first order condition, we obtain that $14c^2 + c - 26 = 0$, which implies $c^* = 1.328$. The price paid by Caltech is obtained plugging the value of c into the expression $x = B \cdot c + D \cdot c^2$. After doing that, the value $x = 27.32$ is obtained. Such value represents the entire amount of money paid by Caltech for the pipe.

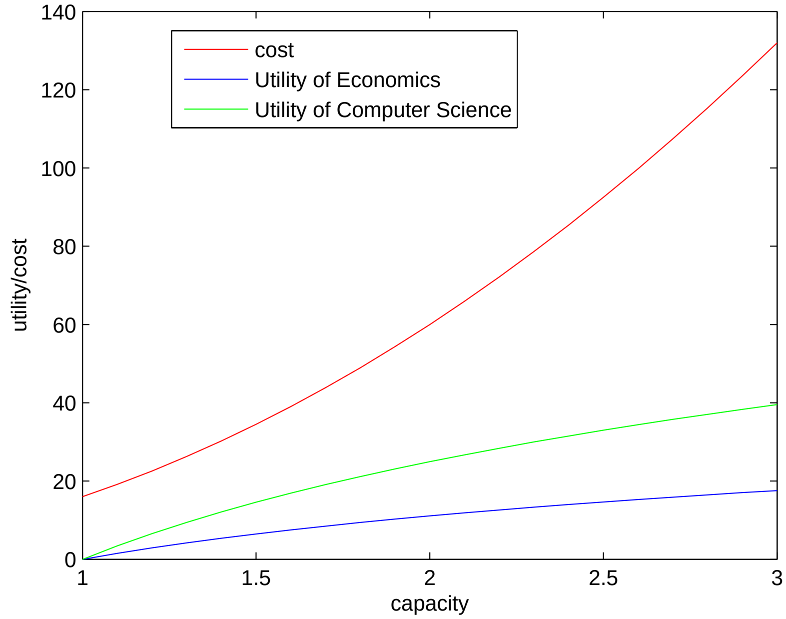

Figure 1: Cost and Utility functions

(c) Yes, there is a difference. A larger amount of capacity is paid by CalTech when trying to maximize its total utility minus costs. In part A the Economics Department is not paying any money and it is benefiting from knowing that Computer Science will buy enough capacity for it too. In economic terms, we say that the Economic department free rides.

(d) The Lindhal prices are $q_1 = a_1/c^*$ and $a_2 = c^*$. After the appropriate subsitution, we obtain $q_1 = 12.053$ and $q_2 = 27.18$. It can be verified that this fulfills the equation $q_1 + q_2 = B + 2DC$, where the right hand side is the derivative of the cost function and B and D are known parameters of the problem. So, if Caltech decides to charge the price q_i for Department i, then the optimal capacity to buy for both department will be exactly c^*.