CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. (a) Tree for this game

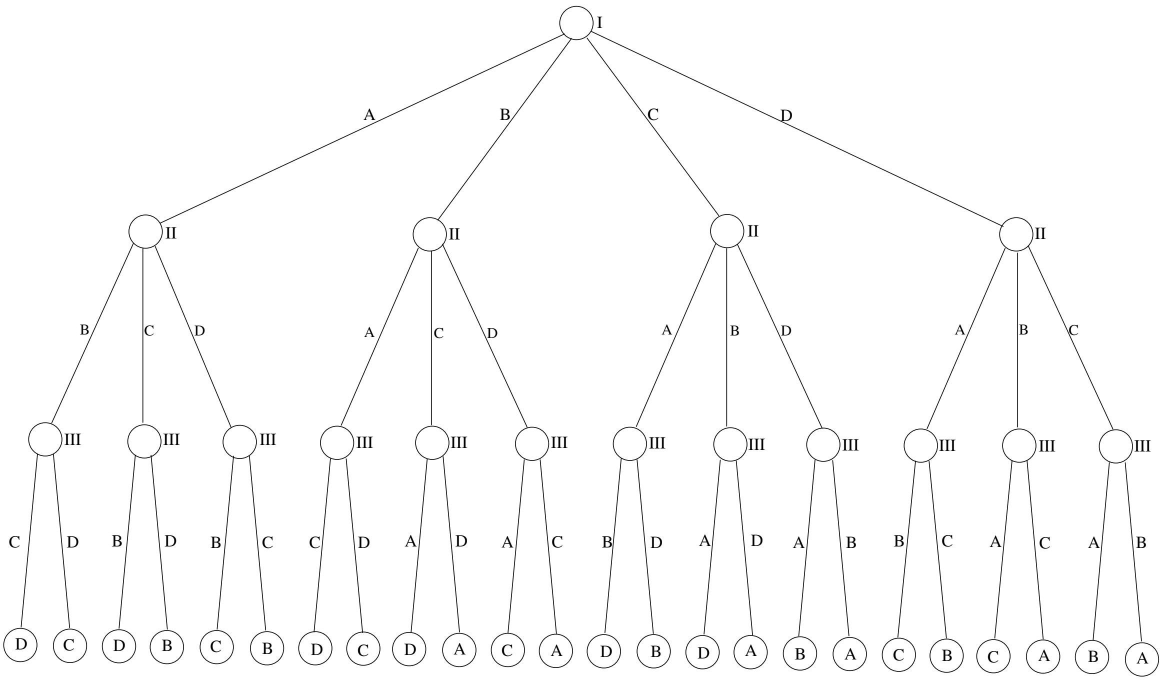

Figure 1: Tree for the game in exercise 1(a)

(b) (a) Player 1 has 4 strategies.
 (b) Player 2 has 3^4 strategies.
 (c) Player 3 has 2^{12} strategies.

(c) We use the mechanism of backward induction to decide that, as follows. The arrow denotes that the player vetoes the candidate who labels the edge, see Figure 2.

The conclusion is that C wins if a subgame perfect equilibrium is used.
A subgame perfect equilibrium is the following:

I: A

II: (B, A, A, A)

III: $(D, B, B, D, A, A, B, A, B, B, A, B)$

(d) Consider the following subgame:

I: D

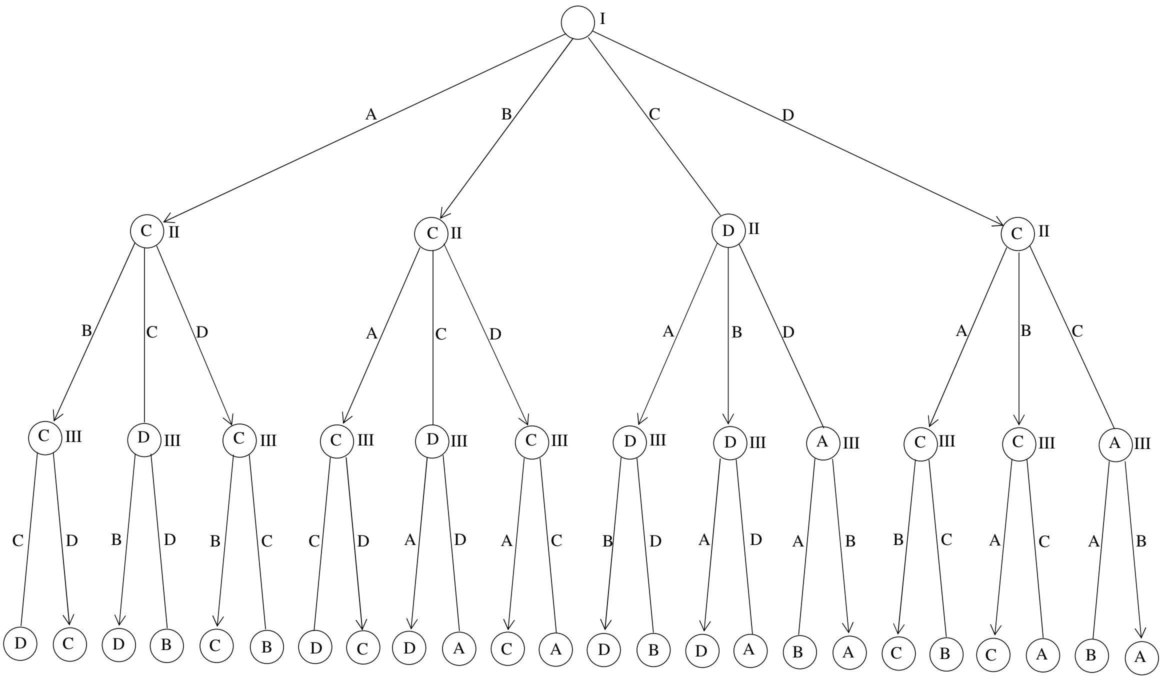

Figure 2: Tree for the game in exercise 1(c)

II: (C, A, B, A)

III: $(D, B, B, D, A, A, B, A, B, B, A, B)$

This is clearly not a subgame perfect equlibrium. In fact, if player 1 would play A, then player 2 would play C and player 3 would play B. So, the final outcome would be D.

A better move for player 2 would have been to play D instead of C. In fact, knowing that player 3 would have played B, this would have given an outcome of C.

2. (a) See Figure 3, where (i, j) is such that i denotes the money of player 1 and j denotes the money of player 2.

   (b) She will not find the incumbent monopolist credible because fighting will be worse than acquiescencing for player 1.

   (c) If they both act rationally, then the following game will be played:

      - Since player 2 thinks that player 1 acts rationaly, then player 2 will enter the market because he would get more then staying out and player 1 will act rationally, which means that he will acquiesce. In fact, if he acquiesces the outcome for player 1 will be better then if he decides to fight. The conclusion, if they both act rationally, then:
         - player 1 acquiesces
         - player 2 enters the market.

3. (a) See Figure 4

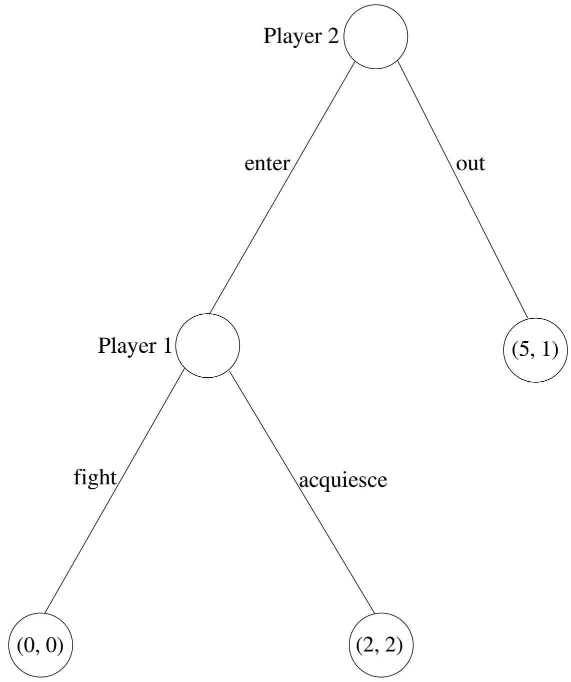

Figure 3: Tree for the game in Exercise 2(a)

(b) Applying backward induction get we obtain the tree in Figure 5. The arrow in the tree indicates the choice of the player, the outcome for a given choice is represented by a pair (a, b) inside the circle. It clearly appears that there is only one subgame perfect equilibrium which is given by

I: $invest$

II: $(out, enter)$

III: $(-, fight, -, acquiesce)$

The symbol $-$ denotes the absence of action.

(c) A game theorist might replay that investing is like an irreversible commitment for player 1. Player 2 now knows that player 1 would be better off fighting than acquiescencing if he enters the market. So it would be better for player 2 to stay out of the market and get an outcome of 1 rather than entering the market and getting an outcome of 0.

player 2 is that player 2 is rational. In this case when player 2 is not rational, then he might choose to enter the market and then it makes sense for player 1 to fight.

4. (a) Yes, $(1, 1, 1)$ is such a strategy profile. Suppose we have the profile $(1, 1, 1)$. If player i increases his number to $x > 1$, then he will have a smaller gain for the following reason. The fraction $\frac{2}{3}$ of the average is:

$$\frac{2}{3}\left(\frac{1+1+x}{3}\right) = \frac{4+2x}{9} \tag{1}$$

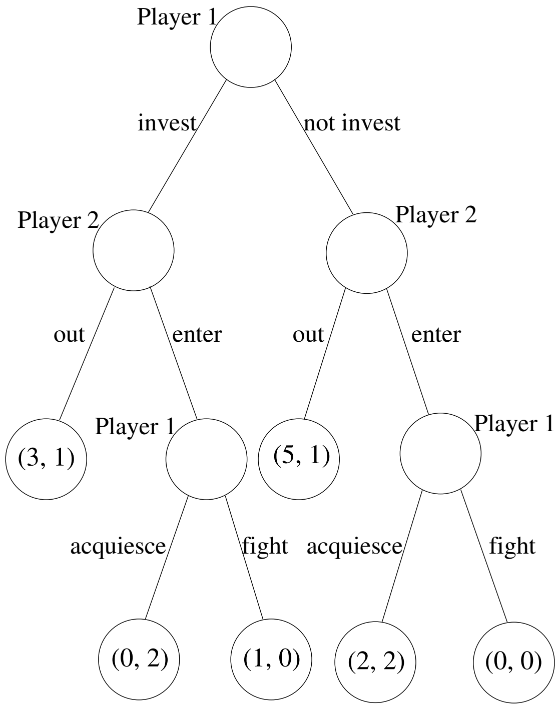

Figure 4: Tree for the game in Exercise 3(a)

It is easy to check that the absolute value of the difference $|1 - \frac{2x-5}{9}|$ is smaller than the absolute value of the difference $|1 - \frac{7x-4}{9}|$ for any $x > 1$. Hence, player 1 will be farther than any other player from the expression in equation 1. The conclusion is that $1, 1, 1$ is a Nash equilibrium.

(b) No, there is no other profile which is a Nash equilibrium. We will prove the stronger statement that $(1, 1, 1)$ is the unique Nash equilibrium of the game. We start proving that (h, g, k), is not a Nash equilibrium whenever the three numbers are not the same. This will imply that if there is a Nash equilibrium, it must be of the form (k, k, k) for some positive $k \geq 1$. Suppose by contradiction that this is not true and let $v = \max\{h, g, k\}$ be the number played by Player 1. This can be done without loss of generality since the proof is symmetric for every player. It is obvious that Player 1 will not get any money since there will be at least another player who is closer than him to $\frac{2}{3}$ of the average. Let him be Player 2 and let q be the number he has chosen. Suppose now that Player 1 moves his choice to q. Then his outcome will certainly be greater than 0 (he will certainly split the money with Player 2 and possibly with Player 3). The conclusion is that (h, g, k) can not be a Nash equilibrium. So far, we have proven that any Nash equilibrium will be of the form (k, k, k), for $k \geq 1$. The last part of the proof shows that the only possibility for a Nash equilibrium is $(k, k, k) = (1, 1, 1)$. Suppose by contradiction that (k, k, k) for $k \geq 2$ is a Nash equilibrium. If player i moves his choice to 1, while the other players do not change their choices, then player i will be the only player to get the outcome, i.e. he will not have to split it with anyone else. For that we need to show that the inequality 2

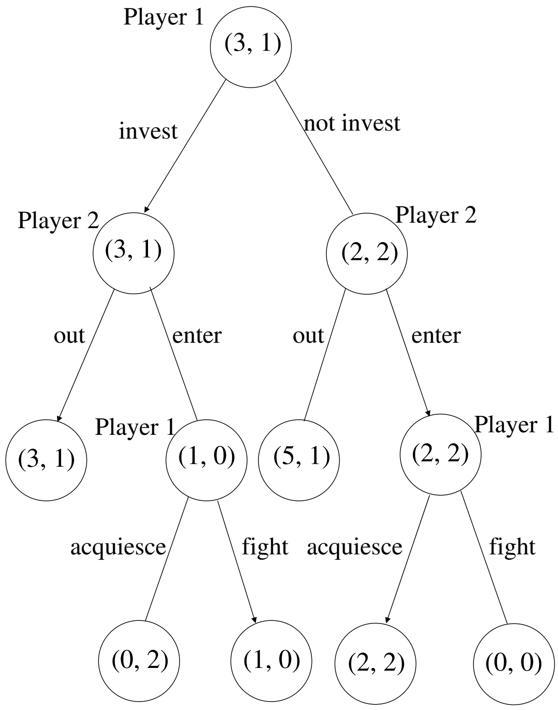

Figure 5: Tree for the game in Exercise 3(b)

holds:

$$\left|1 - \frac{2}{3}\left(\frac{2k+1}{3}\right)\right| < \left|k - 2\left(\frac{2k+1}{9}\right)\right| \tag{2}$$

Inequality 2 is true iff

$$\left|\left(\frac{9-4k-2}{9}\right)\right| < \left|\left(\frac{5k-2}{9}\right)\right| \tag{3}$$

It can be checked that the above expression is always true for any $k > 1$ (take the square of the left hand side and the right hand side to verify it).

The overall conclusion is that $(1, 1, 1)$ is the only Nash equilibrium of the game.

(c) The only rationalizable strategy for each player is to play 1.

Proof. Let s be the number played by Player 1. Consider the function

$$\left|s - \frac{2}{3}\frac{s+h+k}{3}\right| \tag{4}$$

Fixing h and k, we have that this function is increasing for $s \geq (h+k)/7$. Consequently, player 1 will never play more than $(K+K)/7$, where K is the largest number which can be used. Then, player 2 will not play anything bigger than $\frac{K+\frac{K+K}{7}}{7}$. We can continue iterating in this way until we get that no player will ever use a number l, $l > 1$. This proves that $(1, 1, 1)$ is the only rationalizable strategy.