CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. (a) Let n be the number of students and m the number of houses. We try to formulate the optimization problem as follows.

$$
\begin{array}{lll}
\text{Maximize} & \sum_{k=1}^{n}\sum_{h=1}^{m} c_{k,h}^T x_{k,h} & \\
\text{subject to:} & & \\
 & \sum_{h=1}^{m} x_{h,k} = 1 & \forall k \\
 & \sum_{k=1}^{n} x_{h,k} \leq v_h & \forall h \\
 & x_{h,k} \geq 0 & \forall k,h
\end{array}
$$

where $c = (c_{k,h})$ is a column vector consisting of $n \cdot m$ components, with the component $c_{k,h}$ preceding $c_{i,j}$ if $k \leq i$ or $k = i$ and $h <= j$. The (k,h)-th entry denotes the happiness of student k in house h.

The optimization variable x is the column vector consisting of $n \cdot m$ components, with the component $x_{k,h}$ preceding $x_{i,j}$ if $k \leq i$ or $k = i$ and $h <= j$. The (k,h)-th entry of the vector *should* represent the assignment of student k to house h.

(b) The problem cannot be formulated in standard form because the constraint $x_{k,h} \geq 0$ is too loose. The variable $x_{k,h}$ must be a binary variable denoting whether or not student k has been assigned to house h. Therefore, a solution x with some component $x_{k,h} = p$, $p \neq 0, 1$, would be senseless. In order to obtain the desired result, we must impose the additional constraint that x is an integer vector whose components can only take values 0 and 1. Notice that this additional constraint makes the problem nonconvex since $x_{h,k} \in \{0, 1\}$ for every h, k is a nonconvex constraint. Under this assumption the simplex method is no longer guaranteed to return the optimal solution.

Remark: For this special problem the solution obtained without imposing the integer constraint on the variables would have been the same as the solution obtained imposing the integer constraint! However, in general an integer programming problem has a different solution than a linear programming problem.

2. Consider the following optimization problem:

$$
\begin{array}{llllll}
\text{Maximize} & z & & & & \\
\text{where} & 9x_0 & +7x_1 & +6x_2 & +9x_3 & = z \\
\text{subject to:} & & & & & \\
 & 4x_0 & +2x_1 & +1x_2 & +3x_3 & \leq 8 \\
 & 1x_0 & +1x_1 & +3x_2 & +1x_3 & \leq 9 \\
 & x_0, & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$

Convert it into the canonical form adding the two slack variables s_0, s_1.

$$
\begin{array}{lllllllll}
\text{Maximize} & z \\
\text{where} & +9x_0 & +7x_1 & +6x_2 & +9x_3 & +0s_0 & +0s_1 & = z \\
\text{subject to:} \\
 & +4x_0 & +2x_1 & +1x_2 & +3x_3 & +1s_0 & +0s_1 & = 8 \\
 & +1x_0 & +1x_1 & +3x_2 & +1x_3 & +0s_0 & +1s_1 & = 9 \\
 & x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}
$$

Find non–basic and basic variables. Non–basic variables are x_0, x_1, x_2, x_3 and basic variables are s_0, s_1. The corresponding basic feasible solution is $x_0 = x_1 = x_2 = x_3 = 0, s_0 = 8, s_1 = 9$ and the value of z is 0.

Among all variables with positive coefficients in z, consider x_0. Increase x_0 until the first between s_0, s_1 decreases in value to 0. We have that when x_0 increases to 2, s_0 decreases to 0 while s_1 is still positive. Therefore, in the next step, non–basic variables are x_1, x_2, x_3, s_0 and basic variables are x_0, s_1. The corresponding basic feasible solution is $s_0 = x_1 = x_2 = x_3 = 0, x_0 = 2, s_1 = 7$ and the value of z is 18.

Convert the problem to the canonical form for new basic variables by pivoting on $4x_0$, i.e. the element in the column of the incoming basic variable (column 1) and in the row of the outgoing basic variable (row 1).

$$
\begin{array}{lllllllll}
\text{Maximize} & z \\
\text{where} & +9x_0 & +7x_1 & +6x_2 & +9x_3 & +0s_0 & +0s_1 & = z \\
\text{subject to:} \\
 & \boxed{+4x_0} & +2x_1 & +1x_2 & +3x_3 & +1s_0 & +0s_1 & = 8 \\
 & +1x_0 & +1x_1 & +3x_2 & +1x_3 & +0s_0 & +1s_1 & = 9 \\
 & x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}
$$

Divide the first constraint by 4. Substract the first constraint from the second constraint. Substract 9 times the first constraint from the objective function. Obtaining

$$
\begin{array}{lllllllll}
\text{Maximize} & z \\
\text{where} & +0x_0 & +\frac{5}{2}x_1 & +\frac{15}{4}x_2 & +\frac{9}{4}x_3 & -\frac{9}{4}s_0 & +0s_1 & = z - 18 \\
\text{subject to:} \\
 & +1x_0 & +\frac{1}{2}x_1 & +\frac{1}{4}x_2 & +\frac{3}{4}x_3 & +\frac{1}{4}s_0 & +0s_1 & = 2 \\
 & +0x_0 & +\frac{1}{2}x_1 & +\frac{1}{4}x_2 & +\frac{1}{4}x_3 & -\frac{1}{4}s_0 & +1s_1 & = 7 \\
 & x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}
$$

Among all variables with positive coefficients in z, consider x_1. Increase x_1 until the first between x_0, s_1 decreases in value to 0. We have that when x_1 increases to 4, x_0 decreases to 0 while s_1 is still positive. Therefore, in the next step, non–basic variable are x_0, x_2, x_3, s_0 and basic variables are x_1, s_1. The corresponding basic feasible solution is $s_0 = x_0 = x_2 = x_3 = 0, x_1 = 4, s_1 = 5$ and the value of z is 28.

Convert the problem to the canonical form for the new basic variables by pivoting on $\frac{1}{2}x_1$, i.e. the element in the column of the incoming basic variable (column 2) and the row of the outgoing basic variable (row 1).

$$\begin{array}{llllllll}
\text{Maximize} & z \\
\text{where} & +0x_0 & +\frac{5}{2}x_1 & +\frac{15}{4}x_2 & +\frac{9}{4}x_3 & -\frac{9}{4}s_0 & +0s_1 & = z - 18 \\
\text{subject to:} \\
& +1x_0 & +\boxed{\frac{1}{2}x_1} & +\frac{1}{4}x_2 & +\frac{3}{4}x_3 & +\frac{1}{4}s_0 & +0s_1 & = 2 \\
& +0x_0 & +\frac{1}{2}x_1 & +\frac{1}{4}x_2 & +\frac{1}{4}x_3 & -\frac{1}{4}s_0 & +1s_1 & = 7 \\
& x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}$$

Substract 5 times the first constraint from the objective function. Substract the first constraint from the second constraint. Multiply the first constraint by 2. The following result is obtained:

$$\begin{array}{llllllll}
\text{Maximize} & z \\
\text{where} & -5x_0 & +0x_1 & +\frac{5}{2}x_2 & -\frac{3}{2}x_3 & -1s_0 & +0s_1 & = z - 28 \\
\text{subject to:} \\
& +2x_0 & +1x_1 & +\frac{1}{2}x_2 & +\frac{3}{2}x_3 & +\frac{1}{2}s_0 & +0s_1 & = 4 \\
& -1x_0 & +0x_1 & +\frac{5}{2}x_2 & +\frac{1}{2}x_3 & -\frac{1}{2}s_0 & +1s_1 & = 5 \\
& x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}$$

Among all variables with positive coefficients in z, consider x_2. Increase x_2 until the first between x_1, s_1 decreases in value to 0. We have that when x_2 increases to 2, s_1 decreases to 0 while x_1 is still positive. Therefore, in the next step, the non–basic variables are x_0, x_3, s_0, s_1 and the basic variables are x_1, x_2. The corresponding basic feasible solution is $s_0 = s_1 = x_0 = x_3 = 0, x_2 = 2, x_1 = 3$ and the value of z is 33.

Convert the problem to the canonical form for the new basic variables by pivoting on $\frac{5}{2}x_2$, i.e. the element in the column of the incoming basic variable (column 2) and in the row of the outgoing basic variable (row 2).

$$\begin{array}{llllllll}
\text{Maximize} & z \\
\text{where} & -5x_0 & +0x_1 & +\frac{5}{2}x_2 & -\frac{3}{2}x_3 & -1s_0 & +0s_1 & = z - 28 \\
\text{subject to:} \\
& +2x_0 & +1x_1 & +\frac{1}{2}x_2 & +\frac{3}{2}x_3 & +\frac{1}{2}s_0 & +0s_1 & = 4 \\
& -1x_0 & +0x_1 & +\boxed{\frac{5}{2}x_2} & +\frac{1}{2}x_3 & -\frac{1}{2}s_0 & +1s_1 & = 5 \\
& x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}$$

Substract the first constraint from the objective function. Multiply the second constraint by $\frac{2}{5}$. Substract $\frac{1}{2}$ times the second constraint from the first one. The following result is obtained:

$$\begin{array}{llllllll}
\text{Maximize} & z \\
\text{where} & -4x_0 & +0x_1 & +0x_2 & -1x_3 & -\frac{1}{2}s_0 & -1s_1 & = z - 33 \\
\text{subject to:} \\
& +\frac{11}{5}x_0 & +1x_1 & +0x_2 & +\frac{8}{5}x_3 & +\frac{3}{5}s_0 & -\frac{1}{5}s_1 & = 3 \\
& -\frac{2}{5}x_0 & +0x_1 & +1x_2 & +\frac{1}{5}x_3 & -\frac{1}{5}s_0 & -\frac{2}{5}s_1 & = 2 \\
& +x_0, & x_1, & x_2, & x_3, & s_0, & s_1 & \geq 0
\end{array}$$

Now, the procedure ends because there are no positive coefficients and returns $z = 33$.

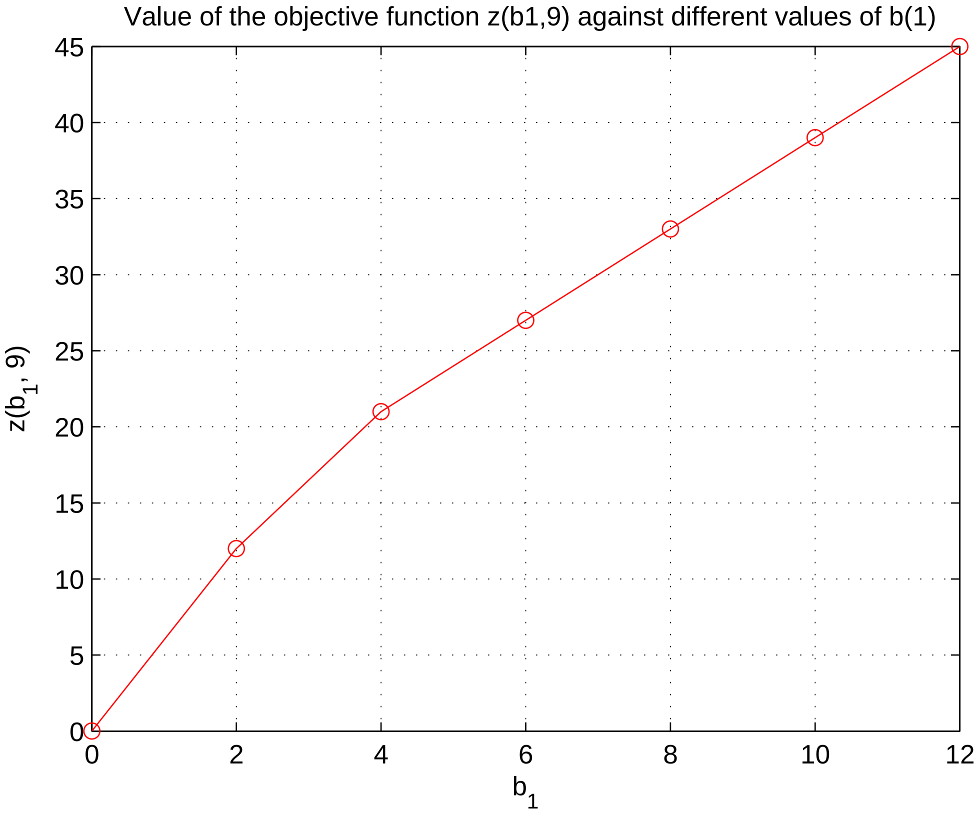
3. (a) Matlab program

```
function mylinprog()

C = [9; 7; 6; 9];         % coefficients of the objective function
A = [4 2 1 3; 1 1 3 1]; % matrix

for i = 0:2:12
    b=[i; 9]; % constraint vector
    [x, val] = linprog(-C, A, b,  [], [], [0 0 0 0], []);
    solmatrix((i/2)+1, :) = x';
    % sol matrix contains the solution vectors.
    % The first row contains the solution when b1=0.
    % The second row contains the solution when b1=2.
    Fval((i/2)+1) = -val;
    % Fval is a vector whose i-th entry is the values of z
    % when b1 = 2*(i-1)
end
plot([0:2:12], Fval, '-or');
grid;
xlabel('b_1');
ylabel('z = (b_1, 9)');
title('Value of the objective function z(b1,9) against different values of b(1)');
```

The solution vector $z(b_1, 9)$, $b_1 = [0, 2, 4, 6, 8, 10, 12]$ is $z(b_1, 9) = [0, 12, 21, 27, 33, 39, 45]$. The plot is shown in figure 3.

We conclude with two observations. The value $z(8, 9)$ is the same that we have found applying the simplex method in exercise 2, i.e. $z = 33$. You may have noticed that the function z plotted is concave, nondecreasing and piecewise linear. We have proved in homework 1 that it is concave and nondecreasing for every convex problem, so this does not come as a surprise. Moreover, if the program is linear, it can be shown that the function z is always piecewise linear.

(b) The dual variable u_1, representing the price of resource b_1 at each point in the graph, is the tangent of the curve at that point. It can be calculated from the equation $z = u \cdot b$. Since the value of z and b_1 are known, it is possible to obtain u applying this formula (see also slides on the web page of the course). The value u_1 represents the additional profit when the amount of resource b_1 is increased. The price is not increasing as b_1 increases since this is diminishing marginal returns. As b_1 tends to infinity the price should drop to zero. This is because in the limit, b_1 is no longer a scarce resource and the profit will be only determined by the other constraint b_2.


Value of the objective function z(b1,9) against different values of b(1)
45
40
35
30
25
20
15
10
5
0
0
2
4
6
8
10
12
b1
z(b1, 9)