CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1.

2. (a) Let $\boldsymbol{x} \in H$ and $\boldsymbol{y} \in H$ be two generic tuples in H. The objective is to prove that for any $a \leq 1$, we have that $\boldsymbol{z} = a \cdot \boldsymbol{x} + (1-a) \cdot \boldsymbol{y}$ is a tuple in H. By definition of H this is true iff $g(\boldsymbol{z}) \leq d$. Due to that g is convex by hypothesis, we have that

$$g(\boldsymbol{z}) = g(a \cdot \boldsymbol{x} + (1-a) \cdot \boldsymbol{y}) \leq a \cdot g(\boldsymbol{x}) + (1-a) \cdot g(\boldsymbol{y}) \tag{1}$$

Since $a \leq 1$ then

$$a \cdot g(\boldsymbol{x}) + (1-a) \cdot g(\boldsymbol{y}) \leq \max\{g(\boldsymbol{x}), g(\boldsymbol{y})\} \leq d \tag{2}$$

Combining 1 and 2 we can conclude that $g(\boldsymbol{z}) \leq d$. This ends the proof.

(b) Suppose by contradiction that H is not closed. Then there exists $\boldsymbol{y} \in \mathrm{dom}(g)$ such that $g(\boldsymbol{y}) > d$ and a sequence of tuples $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ such that

- $\boldsymbol{x}_i \in H$
- $\lim_{i \to \infty} \boldsymbol{x}_i = \boldsymbol{y}$

Since the function g is convex by hypothesis, then g is also continuous in all domain. Recollect the continuous function theorem for sequences stating: If a_n is a sequence of real numbers, and if $\lim_{n\to\infty} a_n = L$, then the $\lim_{n\to\infty} f(a_n) = f(L)$ if f is a function that is continuous at L and defined at all a_n. It holds:

$$\lim_{n\to\infty} g(\boldsymbol{x}_n) = g(\lim_{n\to\infty} \boldsymbol{x}_n) \tag{3}$$

Since $\boldsymbol{x}_i \in H$ then $\lim_{n\to\infty} g(\boldsymbol{x}_n) \leq d$. However $g(\lim_{n\to\infty} \boldsymbol{x}_n) = g(\boldsymbol{y}) > d$. Hence, a contradiction has been obtained and consequenly H must be closed.

3. (a) We start showing that the following statement holds: If for all j we have $g_j(\boldsymbol{x}) \leq b_j$ and $g_j(\boldsymbol{x}) \leq q_j$ for some fixed $\boldsymbol{x}$, then $g_j(\boldsymbol{x}) \leq a \cdot b_j + (1-a) \cdot q_j$ for any $a \leq 1$. Let $\boldsymbol{x} = \boldsymbol{x}_0$ a tuple which verifies the hypothesis of the statement. Due to the previous exercise we know that $H_j = \{\boldsymbol{x} : g_j(\boldsymbol{x}) \leq b_j\}$ is a convex set. Therefore we can find $\boldsymbol{v}, \boldsymbol{w} \in H_j$ such that $\boldsymbol{x}_0 = a \cdot \boldsymbol{v} + (1-a) \cdot \boldsymbol{w}$ for some $a < 1$, $\boldsymbol{v}, \boldsymbol{w} \in H_j$. Due to the assumption that g_j is convex, we have that

$$g_j(a \cdot \boldsymbol{v} + (1-a) \cdot \boldsymbol{w}) \leq a \cdot g_j(\boldsymbol{v}) + (1-a) \cdot g_j(\boldsymbol{w}) \tag{4}$$

By hypothesis of the statement $g_j(\boldsymbol{v}) \leq b_j$ and $g_j(\boldsymbol{w}) \leq q_j$, thus, we have that

$$a \cdot g_j(\boldsymbol{v}) + (1-a) \cdot g_j(\boldsymbol{w}) \leq a \cdot b_j + (1-a) \cdot q_j \tag{5}$$

This ends the proof of the statement. It can now be easily concluded that for any $a < 1$ and for any pair $\boldsymbol{b}, \boldsymbol{q}$ of vectors

$$z(a \cdot \boldsymbol{b} + (1-a) \cdot \boldsymbol{q}) \geq \max\{z(\boldsymbol{b}), z(\boldsymbol{q})\} \geq a \cdot z(\boldsymbol{b}) + (1-a) \cdot z(\boldsymbol{q}) \tag{6}$$

The first part of inequality 6 follows because every feasible solution for $z(\boldsymbol{b})$ and $z(\boldsymbol{q})$ is also a feasible solution for $z(a \cdot \boldsymbol{b} + (1-a) \cdot \boldsymbol{q})$ due to the statement. Since $a < 1$ it is easy to see that also the second part of the inequality holds.

Hence, we have proven that z is a concave function.

(b) z is in general nonlinear. In order to convince yourself, consider the function $z(\boldsymbol{b})$, where $\boldsymbol{b} = (b_1, b_2)$ is two dimensional vector defined as:

$$z(\boldsymbol{b}) = \max -x^2$$
$$\text{subject to: } -x \leq b_1, x \leq b_2$$

Notice that $-x^2$ is a concave function and the two constraints in the problem are linear (therefore convex).

Let $\boldsymbol{b} = (-2, 3)$ and $\boldsymbol{q} = (-3, 4)$. We then have that $z(\boldsymbol{b}) = -4$ and $z(\boldsymbol{q}) = -9$. We also have that $z(\boldsymbol{b} + \boldsymbol{q}) = 25$, where $\boldsymbol{b} + \boldsymbol{q} = (-5, 7)$. Therefore $z(\boldsymbol{b} + \boldsymbol{q}) \neq z(\boldsymbol{b}) + z(\boldsymbol{q})$ which means that z is nonlinear.

(c) The function z is monotonic non decreasing, i.e. if $\boldsymbol{b} \geq \boldsymbol{q}$, then $z(\boldsymbol{b}) \geq z(\boldsymbol{q})$. If the vector $\boldsymbol{x}$ maximizes f when the vector of constraints is $\boldsymbol{q}$ (i.e. $g_j(\boldsymbol{x}) \leq q_j$), then the same vector $\boldsymbol{x}$ will be a feasible solution for the maximization problem when the vector of constraints is $\boldsymbol{b}$ (in fact $g_j(\boldsymbol{x}) \leq q_j \leq b_j$). Hence, $z(\boldsymbol{b}) \geq z(\boldsymbol{q})$.