CALIFORNIA INSTITUTE OF TECHNOLOGY
Selected Topics in Computer Science and Economics

CS/EC/101b

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1. I have two coins, let's call them coin A and coin B. A toss of coin A yields a head with probability p, and a toss of coin B yields a head with probability q independent of all other tosses. I pick one of the two coins, A or B, randomly and put the picked coin in my pocket. (I give the other coin to charity and it plays no further role in this story.) Let the probability that I pick coin A be u (and hence the probability that I pick coin B is $1 - u$).

   (a) I take the coin out of my pocket and toss it a certain number of times and I get h heads and t tosses (from a total of $h + t$ tosses). What is the conditional probability that the coin that I'm tossing is coin A given this experimental result?

   (b) Consider the special case where $q = 1 - p$ and $u = 0.5$. Assume $p > 0.5$. An experiment is conducted in which the coin is tossed a certain number of times, and the numbers of heads and tails that occurred are noted. Let h and t be the numbers of heads and tails, respectively. You are given the value x defined as:

   $$x = h - t$$

   For example, if in 20 tosses there are 15 heads and 5 tails you are told that x is +10, but you are not given the numbers 20, 15 or 5. You are offered the following opportunity. The coin will be tossed again and if you call the outcome correctly (either heads or tails) you get paid $100. If you call incorrectly you get nothing.

   You are offered the following additional information for 1: You will be told exactly how many heads and tails occurred in the experiment (and not merely the difference). For example, if the coin was tossed 20 times and there were 15 heads and 5 tails you will be given the information that $h = 15$ and $t = 5$.

   i. Would you buy this information?
   ii. Explain your answer.
   iii. What strategy will you follow? For example, if you *don't* buy the information then under what circumstances will you call heads? If you *do* buy the information, under what circumstances will you call heads?
   iv. Now consider the case where $q \neq 1 - p$. For example, suppose $p = 0.8$ and $q = 0.4$. Assume that u remains 0.5. For this case, would your answers to any of the previous three parts change? First just answer yes or no. Then explain your answer.
   v. Now consider the case where $q = 1 - p$, but $u \neq 0.5$. For example, assume $u = 0.8$. For this case, would your answers to any of the first three parts change? First just answer yes or no. Then explain your answer.

2. The Economics and Computer Science department at Caltech are buying an Internet pipe that the two departments will share. The utility of a pipe of capacity C for department j, ignoring costs, is:

$$utility[j] = a_j * ln(C)$$

THe cost of a pipe with capacity C is:

$$B * C + D * C^2$$

where B and$D$ are given positive constants.

Note that the utility of each department depends *only on the capacity of the pipe* (and is independent of the other utility or activity of the other department given the pipe capacity).

Each department must offer to pay a certain amount of money to buy the pipe; let the amount of money paid by department j be \$ x_j. Then the total amount of dollars to buy the pipe is $x_1 + x_2$, and of course the capacity C must be such that the cost of the pipe is equal to the money paid for the pipe, i.e.,

$$x_1 + x_2 = B * C + D * C^2$$

The utility of department j including costs is called $totalUtility[j]$ and defined to be the utility of the pipe excluding costs $-$ the price of the pipe in dollars:

$$totalUtility[j] = a_j * ln(C) - x_j$$

The only decision for department j is to select x_j. Each department makes its decision to maximize its total utility.

Assume that all players know everything about the game. Everybody knows all the parameters and functions.

Consider the case where $B = 2$ and $D = 14$. Let $a_1 = 16$ and let $a_2 = 36$. (You can assume that the CS department has index 2 and the Economics department has index 1 because CS wants large capacity more than Economics does. Of course it doesn't matter which department is 1 and which is 2.)

(a) What are the optimal values of x_1 and x_2?

(b) Assume that the utility for Caltech for a pipe with capacity C, excluding price paid for the pipe is $(a_1 + a_2) * ln(C)$. If Caltech is paying the entire amount for the pipe (as opposed to each of the departments paying for the pipe separately), how much capacity would Caltech buy? Assume that Caltech optimizes its total utility (pipe utility - pipe cost).

(c) Is there a difference between the capacity that Caltech buys optimally and each of the departments buy, i.e., is there a difference between the previous two answers? If so, why?

(d) What are "Lindahl prices"? What can Caltech do to to get each of the departments to pay Lindahl prices and yet get the best utility for Caltech as a whole (i.e., best toatl utility for both departments together)?