The Plan

- Projection Methods & Constraints
  - Classical Methods
  - Symmetric Methods
  - Symplectic Methods
  - Variational Approach

Pendulum Revisited

Pendulum in 2D (m-g-L-1):

\[
\begin{pmatrix}
\dot{q}_x \\
\dot{q}_y
\end{pmatrix} =
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix},
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix} =
\begin{pmatrix}
-\dot{q}_y \lambda \\
-\dot{q}_x \lambda
\end{pmatrix}
\]

\[
\lambda = \frac{||p||^2 - \dot{q}_y}{||\dot{q}||^2}
\]

Start with \( ||\dot{q}||^2 = 1 \) and differentiate.

Apply midpoint and measure \( ||q||^2 \)

\begin{itemize}
  \item What happened to preserving quadratic invariants?
\end{itemize}

Weak Invariants

Suppose we have:

\[
y = f(y)
\]

\[
M = \{ y : g(y) = 0 \}
\]

\[
y_0 \in M \Rightarrow y(t) \in M \forall t
\]

Then:

\[
g'(y)f(y) = 0 \quad \forall y \in M
\]

Midpoint won't preserve this

- Measures \( f \) off of \( M \)

Example: Energy Preservation

Why not use this to preserve energy?

- “Force” explicit Euler?
Example: Energy Preservation

Why not use this to preserve energy

“Force” symplectic Euler

Projection won’t preserve your integrator’s properties!

Symmetric Projection

Can we keep some structure?

Find $\mu$ s.t.

Check this is symmetric!

Reminder: Holonomic Constraints

Have some standard Lagrangian

$\mathcal{L}(q, \dot{q}) = K(q) - W(q)$

Want to enforce $g(q) = 0$

“Holonomic” since $g$ depends only on $q$

Rewrite Lagrangian

$\mathcal{L}(q, \dot{q}, \lambda, \dot{\lambda}) = K(q) - W(q) - g(q)^T \lambda$

Do variations w.r.t $q$ & $\lambda$

Sympctic Euler?

$p$ implicit, $q$ explicit

$p_{n+1} = p_n - h(H_p(p_n) + (\nabla g)^T \lambda_{n+1})$

$q_{n+1} = q_n + h\frac{\partial H_p}{\partial p}(p_{n+1})$

$0 = g(q_{n+1})$

$p$ not in cotangent space!

$p_{n+1} = p_{n+1} + (\nabla g)^T \lambda_{n+1}$

$0 = \nabla g(q_{n+1})H_p(p_{n+1}, q_{n+1})$

This is still symplectic

A Note on Symplecticity

Test it as before and get:

$J(\omega_1, \omega_2) = \omega_1^T J \omega_2$

Take $\omega_1, \omega_2 \in M$

Then remember $(\nabla g)\omega = 0 \ \forall \omega \in M$
**SHAKE**

- 2nd order, symmetric, symplectic
  - Assuming \( H \) separable, \( M \) constant
  - Störmer-Verlet

\[
q_{n+1} - 2q_n + q_{n-1} = -\frac{h^2}{2}M^{-1}(\nabla W(q_n) + (\nabla g(q_n))^T \lambda_n)
\]

\[
g(q_{n+1}) = 0
\]

What happened to \( p \)?
  - Approximate via finite differences
  - Or rewrite as before...

**RATTLE**

\[
p_{n+\frac{1}{2}} = p_n - \frac{h}{2}(\nabla W(q_n) + (\nabla g)^T \lambda_n)
\]

\[
q_{n+1} = q_n + hM^{-1}p_{n+\frac{1}{2}}
\]

\[
g(q_{n+1}) = 0
\]

\[
p_{n+1} = p_{n+\frac{1}{2}} - \frac{h}{2}(\nabla W(q_{n+1}) + (\nabla g(q_{n+1}))^T \lambda_{n+1})
\]

**Variational Integrators**

Augmented discrete Lagrangian

\[
L_d(q_k, q_{k+1}, \lambda_k) = h[K_d(q_k, q_{k+1}) - W(q_k, q_{k+1}) - g(q_k)^T \lambda_k]
\]

Vary \( q_{k+1} \) and \( \lambda_{k+2} \):

\[
D_1 L_d(q_k, q_{k+1}, \lambda_k) + D_2 L_d(q_k, q_{k+1}, \lambda_k) = 0
\]

\[
g(q_{k+1}) = 0
\]

**Pendulum Re-revisited (m-g-L-l)**

\[
L_d(q_k, q_{k+1}, \lambda_k) = h \left[ \frac{1}{2} \left\| \frac{q_{k+1} - q_k}{h} \right\|^2 - \frac{q_k^y + q_{k+1}^y}{2} - \lambda_k(||q_k||^2 - 1) \right]
\]

To the board!

\[
q_k - 2q_{k+1} + q_{k+2} = \frac{h^2}{2} - 2\lambda_{k+1}q_{k+1}
\]

\[
||q_{k+1}||^2 - 1 = 0
\]

First Slide: \( \left( \begin{array}{c} \rho_x \\ \rho_y \end{array} \right) = \left( \begin{array}{c} -q_{k+1}^y \\ 1 - q_{k+1}^y \end{array} \right) \)