Gittins Policy on NBUE + DHR(k) Job Sizes

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Performance Modeling, 2009

Outline

1. Gittins Policy
   - Gittins Index
   - Gittins Policy Application
2. NBUE $+$ DHR(k) Distributions
   - Gittins Reduction to FCFS $+$ FB(θ)
     - Gittins Index Properties
     - Policy Properties
   - Pareto Example

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Gittins Index Motivation

- K-Armed Bandit Problem
- Optimal Blind Scheduling

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Gittins Index Candidates

- Payoff?
  - Costs not accounted for
- Payoff - Investment?
  - Doesn't make sense – Payoff and Investment are not necessarily in the same units
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- Maximal Ratio of Payoff to Investment

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Scheduling View of Gittins Index

- We parameterize the Gittins Index over
  - a, the current age of the job
  - T, the service quota
- We can think of varying T as varying the investment.
- $J(a,T) = \frac{E[\text{Job Completes}|T]}{E[T_{\text{Completion}}|T]} = \frac{\int_0^T f(a+t)dt}{\int_0^T \bar{F}(a+t)}$
- $G(a) = \sup_{T \geq 0} J(a,t)$

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- We usually know the distribution, and can approximate it well after some startup time if not
- Optimal!

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Gittins Index Computation

- Exact
  - To compute $G(a)$ exactly, we have to compute $J(a, T)$ for some T.
  - We need to take the analytic minimum of $J(a, T)$ w/rspt to T.
- Approximation
  - We can approximate $J(a, T)$ easily
  - Optimiztion of a computationally expensive function over the real line...
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Derivative Calculation

- To optimize J, we calculate its derivative
- $\frac{\delta J}{\delta T} = \frac{f(a+T)\int_0^T \bar{F}(a+t)dt + \bar{F}(a+T)\int_0^T f(a+t)dt}{\int_0^T \bar{F}(a+t)dt}$
- If we let h represent the hazard rate of the distribution, we have $\frac{\delta J}{\delta T} = \frac{\bar{F}(a+T)(h(a+T)-J(a,T))}{\int_0^T \bar{F}(a+t)dt}$

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Lemmas

- We introduce the notation T_a to represent the optimal T choice for a job of age a
- We omit the proofs for these Lemmas for time and relevance
  - $\forall a, x : a \leq x < a + T_a, G(a) \leq G(x)$
  - $\forall a : T_a < \infty, G(a + T_a) \leq G(a)$

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Proof Overview

- $T_0 \geq k$
- $\forall a : a < T_0, G(a) \geq G(0)$
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Property I

- Take some $x : 0 < x < k$
- As it has a NBUE head, $H(x) \geq H(0)$
- Converting to J, $J(x, \infty) \geq J(0, \infty)$
- $\frac{\bar{F}(x)}{\int_x^\infty \bar{F}(t)dt} \geq \frac{1}{\int_0^\infty \bar{F}(t)dt}$
- Running math, we get $\frac{1}{\int_0^\infty \bar{F}(t)dt} \geq \frac{F(x)}{\int_0^x \bar{F}(t)dt}$
- Back in index form, this gives $G(0) \geq J(0, x)$
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Property II

See the first lemma. The proof is omitted as it is a sufficiently general result.

Property III

- Setting our derivative to zero, we get the equation $\frac{\bar{F}(a+T)(h(a+T)-J(a,T))}{\int_0^T \bar{F}(a+t)dt} = 0$
- Excluding infinite T, the $\bar{F}$ term will not zero, so we have $h(a+T) = J(a,T)$
- For $a \geq k$, we have the DHR property, so $G(a) = J(a,0) = h(a)$
- We have the DHR property, so $G(a)$ is decreasing for $a \geq k$.

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Property IV

See the second lemma. The proof is omitted as it is a sufficiently general result.

Policy Derivation

- We have $\forall a : a < T_0, G(a) \geq G(0)$ and $\forall T_0 : T_0 < \infty, G(T_0) \leq G(0)$
- So, the Gittins Index passes its starting position at some point.
- We have $\forall a : a > k, G(a)$ is decreasing
- So, the Gittins Index keeps going down after that.
- As we start NBUE, and end with this property, by optimality of Gittins
- FCFS $+$ FB(T_0)
- Additionally, we have the bound $T_0 > k$

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Qualification

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- After k, DHR
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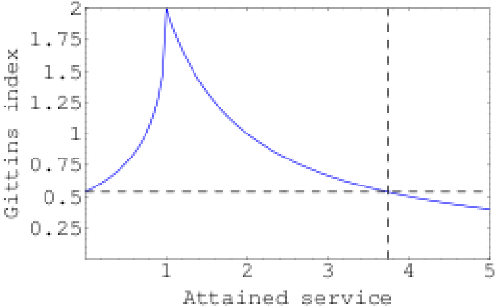
Gittins Index



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- When doing blind scheduling, Gittins Policy is optimal.
- The Gittins Policy is usually intractible.
- In our particular case Gittins reduces to FCFS + FB(T_0) for NBUE + DHR(k).

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